Department of Electrical Engineering
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ELEG 5693 Wireless Communications
Ch. 7 Multicarrier Modulation

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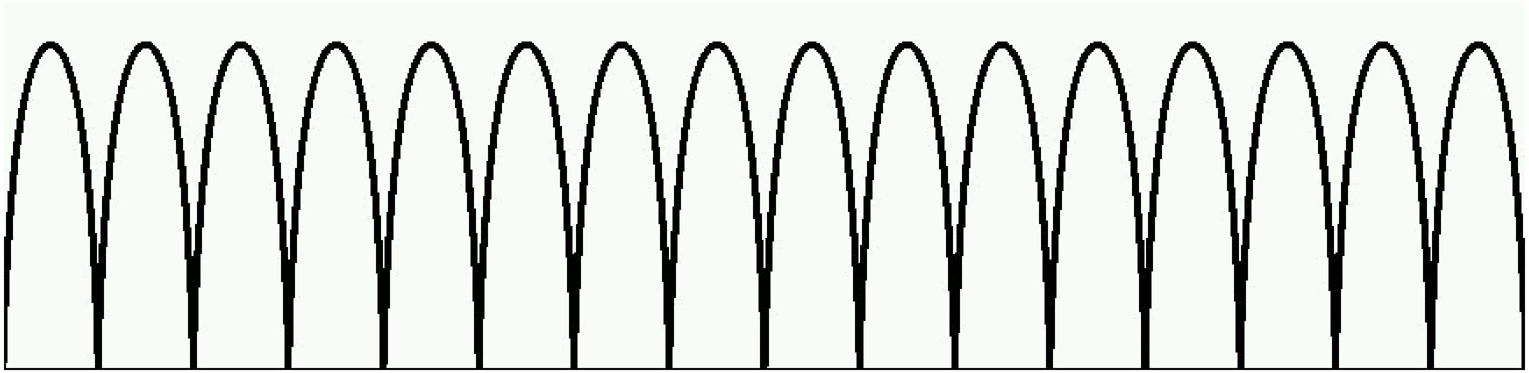
OUTLINE

- **Introduction**
- **OFDM**
- **Challenges in Multicarrier Systems**

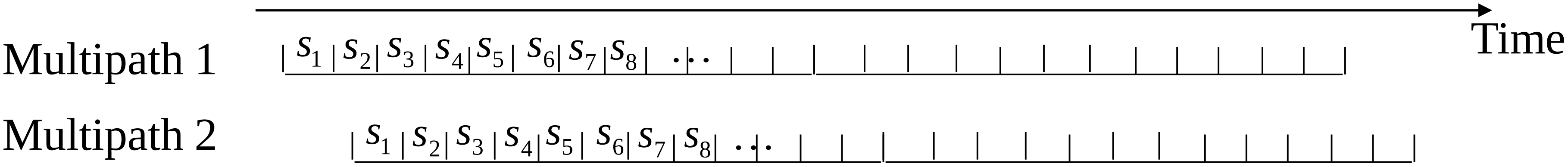
INTRODUCTION

- **Multicarrier Modulation (MCM)**
  - Achieves broadband communication by modulating a large number of narrow-band data streams over closely spaced sub-carriers.
  - Divide a broadband channel into many narrow-band sub-channels in the frequency domain
    - Each sub-channel is called a sub-carrier
  - Divide the transmitted data stream into many sub-streams, and transmit one sub-stream in one sub-channel
  - The data is transmitted over many sub-carrier → multicarrier modulation

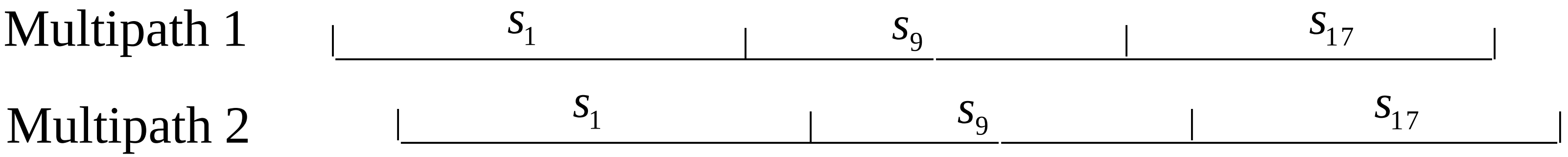


INTRODUCTION

- **Why MCM?**
  - Alleviate the impairments caused by inter-symbol interference (ISI)
  - Single-carrier broadband communication suffers serious ISI



    - s_1 will interfere with s_2 and s_3 ➔ ISI
  - In MCM, each subcarrier delivers a narrow-band signal
    - Narrow-band ➔ longer symbol period



    - Less ISI
    - ISI can be completely removed by using guard interval between adjacent symbols

INTRODUCTION

- **Why MCM (Cont'd)**
  - Original single-carrier system experiences frequency selective fading
    - Data rate >> coherence bandwidth
    - Symbol period << rms delay spread
    - ➔ freqeuncy selective fading
  - MCM: on each subcarrier: data rate decreases, the symbol period increases
    - data rate << coherence bandwidth
    - Symbol period >> rms delay spread
    - ➔ frequency flat fading

    - No equalization is necessary!

INTRODUCTION

- **MCM techniques**
  - Orthogonal frequency division multiplexing (OFDM)
    - 4G cell phone systems
    - Most wireless LANs (IEEE 802.11x)
    - IEEE802.15 UWB
    - Digital audio and video broadcasting
    - The most popular MCM technique
  - Discrete multitone
    - DSL
  - Vector coding
  - …

OUTLINE

- Introduction
- **OFDM**
- Challenges in Multicarrier Systems

OFDM

- **Orthogonal Frequency Division Multiplexing (OFDM)**
  - Orthogonal: the subcarriers are mutually orthogonal → there is no inter-carrier interference (ICI) among the subcarriers
  - The most popular MCM technique
  - Can be implemented with fast Fourier transform (FFT) and inverse fast Fourier transform (IFFT)

OFDM: DFT AND IDFT

- **Review: DFT and IDFT**
  - discrete Fourier transform (DFT)

$$X[i]=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}x[n]e^{-j2\pi ni/N}$$

  - Inverse discrete Fourier transform (IDFT)

$$x[n]=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}X[i]e^{j2\pi ni/N}$$

  - Linear convolution

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

  - Circular convolution

$$y[n]=x[n]\otimes h[n]=\sum_{k=0}^{N-1}h[k]x[n-k]_N$$

$[k]_N = k$ modulo N

OFDM: DFT AND IDFT

- **Linear convolution v.s. circular convolution**
  - Linear convolution

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

$$h[0],h[1],h[2],h[3]$$

$x[3],x[2],x[1],x[0]$	$y[0]=h[0]x[0]$
$x[3],x[2],x[1],x[0]$	$y[1]=h[0]x[1]+h[1]x[0]$
$x[3],x[2],x[1],x[0]$	$y[2]=h[0]x[2]+h[1]x[1]+h[2]x[0]$
$x[3],x[2],x[1],x[0]$	$y[3]=h[0]x[3]+h[1]x[2]+h[2]x[1]+h[3]x[0]$

  - circular convolution

$$y[n]=x[n]\otimes h[n]=\sum_{k=0}^{N-1}h[k]x[n-k]_N$$

$$h[0],h[1],h[2],h[3]$$

$x[0],x[3],x[2],x[1]$	$y[0]=h[0]x[0]+h[1]x[3]+h[2]x[2]+h[3]x[1]$
$x[1],x[0],x[3],x[2]$	$y[1]=h[0]x[1]+h[1]x[0]+h[2]x[3]+h[3]x[2]$
$x[2],x[1],x[0],x[3]$	$y[2]=h[0]x[2]+h[1]x[1]+h[2]x[0]+h[3]x[3]$
$x[3],x[2],x[1],x[0]$	$y[3]=h[0]x[3]+h[1]x[2]+h[2]x[1]+h[3]x[0]$

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OFDM: DFT AND IDFT

- **DFT and Circular Convolution**
  - Circular convolution in the time domain → multiplication in the frequency domain

$$DFT\left\{y[n]=x[n]\otimes h[n]\right\}=X[i]H[i]$$

  - Circular convolution is preferred in discrete time implementation
    - However, practical system has the effect of linear convolution.

OFDM: CYCLIC PREFIX

- **Discrete-time model of a system with frequency selective fading**

$$y[n] = \sum_{k=0}^{\mu} h(k)x[n-k] + z[n]$$

  - Linear convolution
  - Can we convert linear convolution to circular convolution?

$h[0], h[1], h[2], h[3]$	$h[0], h[1], h[2], h[3]$
$x[3], x[2], x[1], x[0], x[3], x[2], x[1]$	$x[0], x[3], x[2], x[1]$
$x[3], x[2], x[1], x[0], x[3], x[2], x[1]$	$x[1], x[0], x[3], x[2]$
$x[3], x[2], x[1], x[0], x[3], x[2], x[1]$	$x[2], x[1], x[0], x[3]$
$x[3], x[2], x[1], x[0], x[3], x[2], x[1]$	$x[3], x[2], x[1], x[0]$

- Linear convolution between $h[0], h[1], h[2], h[3]$ and $x[1], x[2], x[3], x[0], x[1], x[2], x[3]$
- Circular convolution between $h[0], h[1], h[2], h[3]$ and $x[0], x[1], x[2], x[3]$

$$\underline{x[1], x[2], x[3]}, \underset{\uparrow}{x[0]}, x[1], x[2], x[3]$$

circular prefix

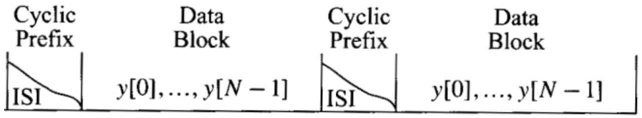
OFDM: CYCLIC PREFIX

- **Cyclic prefix**
  - Circular convolution between **h** (length: $\mu+1$) and **x** can be achieved if
    - 1. Add μ circular prefix to **x,** get $\tilde{\mathbf{x}}$
    - 2. Perform linear convolution between **h** and $\tilde{\mathbf{x}}$

$$\mathbf{x}=\left[x[0],x[1],\cdots,x[N-1]\right] \qquad \tilde{\mathbf{x}}=\left[\underbrace{x[N-\mu],\cdots,x[N-1]}_{\mu\text{ cyclic prefix}},\underbrace{x[0],x[1],\cdots,x[N-1]}_{N\text{ data}}\right]$$

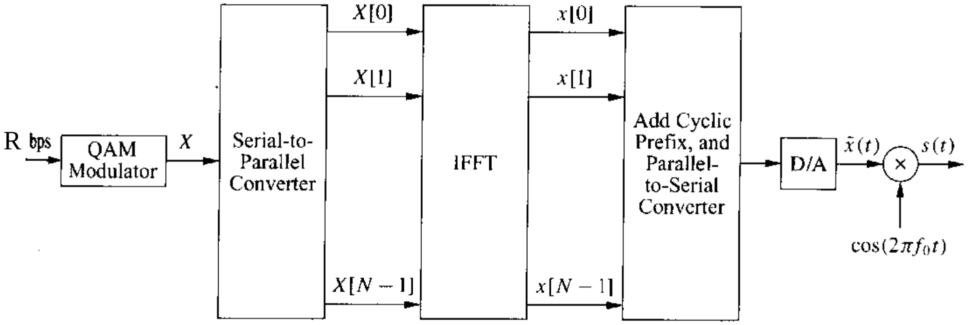
$$y[n]=\sum_{k=0}^{\mu}h[k]\tilde{x}[n-k]=\sum_{k=0}^{\mu}h[k]x\left[n-k\right]_N=h[n]\otimes x[n]$$

  - In a practical system, we can
    - 1. Add circular prefix to the data
    - 2. Transmit the cyclic-prefixed data
    - 3. The channel functions like a linear convolution
    - 4. At the receiver, remove cyclic prefix, we get a circular convolution.



OFDM: TRANSMITTER

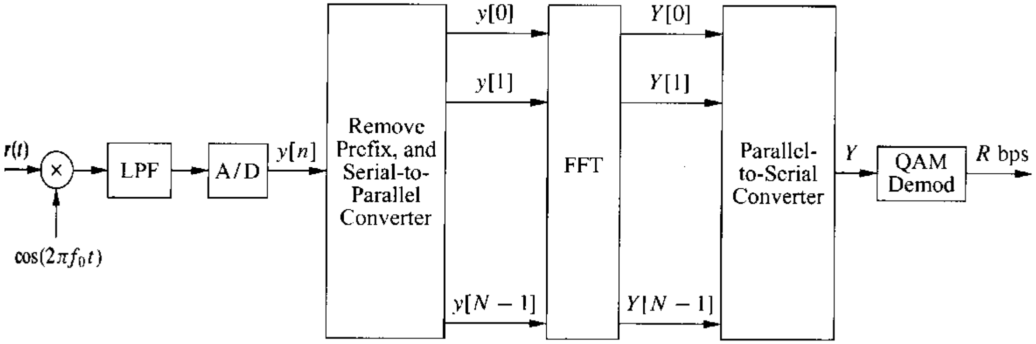
- **OFDM Transmitter**



  - Modulated data: $\mathbf{X} = \left[X[0], X[1], \cdots, X[N-1]\right]$
  - After IFFT: $$x[n] = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} X[i] e^{j2\pi ni/N}$$ $$\mathbf{x} = \left[x[0], x[1], \cdots, x[N-1]\right]$$
  - Add cyclic prefix: $$\tilde{\mathbf{x}} = \left[x[N-\mu], \cdots, x[N-1], x[0], x[1], \cdots, x[N-1]\right]$$

OFDM: RECEIVER

- **Receiver**



Receiver

- Received data sample: $y[n]=\sum_{k=0}^{\mu}h[k]\tilde{x}[n-k]+z[n]=\sum_{k=0}^{\mu}h[k]x[n-k]_N+z[n]=h[n]\otimes x[n]+z[n]$
- After FFT: $Y[i]=H[i]X[i]+Z[i]$ $\quad i=0,\cdots,N-1$

$X[i]$: the data symbol transmitted on the i-th sub-carrier

- $H[i]$: the channel coefficient of the i-th sub-carrier
- The i-th sub-carrier is equivalently modeled as a flat fading channel without ISI
- There is no interference among sub-carriers.

OFDM: MATRIX REPRESENTATION

- **Matrix representation of DFT**
  - Define: DFT matrix

$$\mathbf{Q}=\frac{1}{\sqrt{N}}\begin{bmatrix} e^{-j2\pi\frac{0\cdot 0}{N}} & e^{-j2\pi\frac{0\cdot 1}{N}} & \cdots & e^{-j2\pi\frac{0\cdot(N-1)}{N}} \\ e^{-j2\pi\frac{1\cdot 0}{N}} & e^{-j2\pi\frac{1\cdot 1}{N}} & \cdots & e^{-j2\pi\frac{1\cdot N-1}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j2\pi\frac{(N-1)\cdot 0}{N}} & e^{-j2\pi\frac{(N-1)\cdot 1}{N}} & \cdots & e^{-j2\pi\frac{(N-1)\cdot(N-1)}{N}} \end{bmatrix}$$

  - DFT

$$Y[i]=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} y[n]e^{-j2\pi\frac{ni}{N}}$$

  - DFT in matrix format

$$\begin{bmatrix} Y[0] \\ Y[1] \\ \vdots \\ Y[N-1] \end{bmatrix}=\frac{1}{\sqrt{N}}\begin{bmatrix} e^{-j2\pi\frac{0\cdot 0}{N}} & e^{-j2\pi\frac{0\cdot 1}{N}} & \cdots & e^{-j2\pi\frac{0\cdot(N-1)}{N}} \\ e^{-j2\pi\frac{1\cdot 0}{N}} & e^{-j2\pi\frac{1\cdot 1}{N}} & \cdots & e^{-j2\pi\frac{1\cdot N-1}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j2\pi\frac{(N-1)\cdot 0}{N}} & e^{-j2\pi\frac{(N-1)\cdot 1}{N}} & \cdots & e^{-j2\pi\frac{(N-1)\cdot(N-1)}{N}} \end{bmatrix}\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix}$$

$$\mathbf{Y}=\mathbf{Q}\cdot\mathbf{y}$$

OFDM: MATRIX REPRESENTATION

- **Matrix representation of IDFT**
  - Define: IDFT matrix

$$\mathbf{Q}^H = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j2\pi\frac{0\cdot 0}{N}} & e^{j2\pi\frac{0\cdot 1}{N}} & \cdots & e^{j2\pi\frac{0\cdot(N-1)}{N}} \\ e^{j2\pi\frac{1\cdot 0}{N}} & e^{j2\pi\frac{1\cdot 1}{N}} & \cdots & e^{j2\pi\frac{1\cdot N-1}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j2\pi\frac{(N-1)\cdot 0}{N}} & e^{j2\pi\frac{(N-1)\cdot 1}{N}} & \cdots & e^{j2\pi\frac{(N-1)\cdot(N-1)}{N}} \end{bmatrix}$$

  - IDFT

$$y[n] = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} Y[i] e^{j2\pi\frac{ni}{N}}$$

  - IDFT in matrix format

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j2\pi\frac{0\cdot 0}{N}} & e^{j2\pi\frac{0\cdot 1}{N}} & \cdots & e^{j2\pi\frac{0\cdot(N-1)}{N}} \\ e^{j2\pi\frac{1\cdot 0}{N}} & e^{j2\pi\frac{1\cdot 1}{N}} & \cdots & e^{j2\pi\frac{1\cdot N-1}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j2\pi\frac{(N-1)\cdot 0}{N}} & e^{j2\pi\frac{(N-1)\cdot 1}{N}} & \cdots & e^{j2\pi\frac{(N-1)\cdot(N-1)}{N}} \end{bmatrix} \begin{bmatrix} Y[0] \\ Y[1] \\ \vdots \\ Y[N-1] \end{bmatrix}$$

$$\mathbf{y} = \mathbf{Q}^H \cdot \mathbf{Y}$$

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OFDM: MATRIX REPRESENTATION

- **Matrix representation of OFDM**
  - Transmitter:
    - Data (frequency domain): $\mathbf{X}$
    - IDFT: $\mathbf{x} = \mathbf{Q} \cdot \mathbf{X}$
    - Cyclic prefix: $\tilde{\mathbf{x}}$
  - Channel (time domain):

$$y[n] = \sum_{k=0}^{\mu} h[k]\tilde{x}[n-k] + z[n] = \sum_{k=0}^{\mu} h[k]x[n-k]_N + z[n]$$

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[\mu] \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & \cdots & h[\mu] & h[\mu-1] & \cdots & h[1] \\ h[1] & h[0] & \cdots & h[\mu] & \cdots & h[2] \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ h[\mu] & \cdots & h[1] & h[0] & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & h[\mu] & \cdots & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[\mu] \\ \vdots \\ x[N-1] \end{bmatrix} + \begin{bmatrix} z[0] \\ z[1] \\ \vdots \\ z[\mu] \\ \vdots \\ z[N-1] \end{bmatrix}$$

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$$

H is a cyclic matrix

- Each row is a cyclic shift of the previous row

OFDM: MATRIX REPRESENTATION

- **Matrix representation of OFDM (Cont'd)**
  - Receiver
    - Time domain: $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$
    - Frequency domain: apply DFT at the received signal

$$\mathbf{Y} = \mathbf{Q}\mathbf{y} = \mathbf{Q}\mathbf{H}\mathbf{x} + \mathbf{Q}\mathbf{z} = \mathbf{Q}\mathbf{H}\mathbf{Q}^H \cdot \mathbf{X} + \mathbf{Z} = \mathbf{\Lambda} \cdot \mathbf{X} + \mathbf{Z}$$

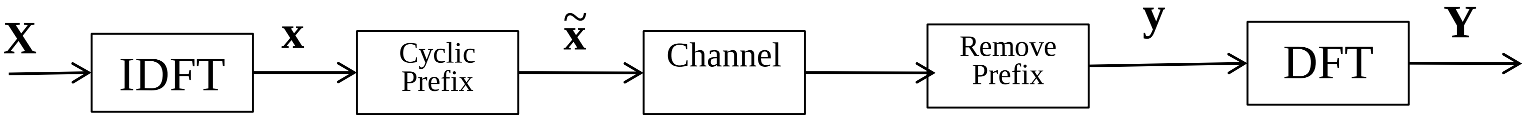
  - Proposition
    - The frequency domain channel matrix, $\mathbf{\Lambda} = \mathbf{Q}\mathbf{H}\mathbf{Q}^H$, is a diagonal matrix, with the i-th diagonal element being

$$H[i] = \sum_{n=0}^{N-1} h[n] e^{-j2\pi n i / N}$$

  - Proof:

OFDM: MATRIX REPRESENTATION

- **Matrix representation of OFDM**
  - Transmitter:
    - Data (frequency domain): $\mathbf{X}$
    - IDFT: $\mathbf{x} = \mathbf{Q} \cdot \mathbf{X}$
    - Cyclic prefix: $\tilde{\mathbf{x}}$
  - Receiver:
    - After removal of cyclic prefix: $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$
    - DFT: $\mathbf{Y} = \mathbf{\Lambda} \cdot \mathbf{X} + \mathbf{Z}$
    - Information on the i-th sub-carrier: $Y[i] = H[i]X[i] + Z[i]$



OFDM: MATRIX REPRESENTATION

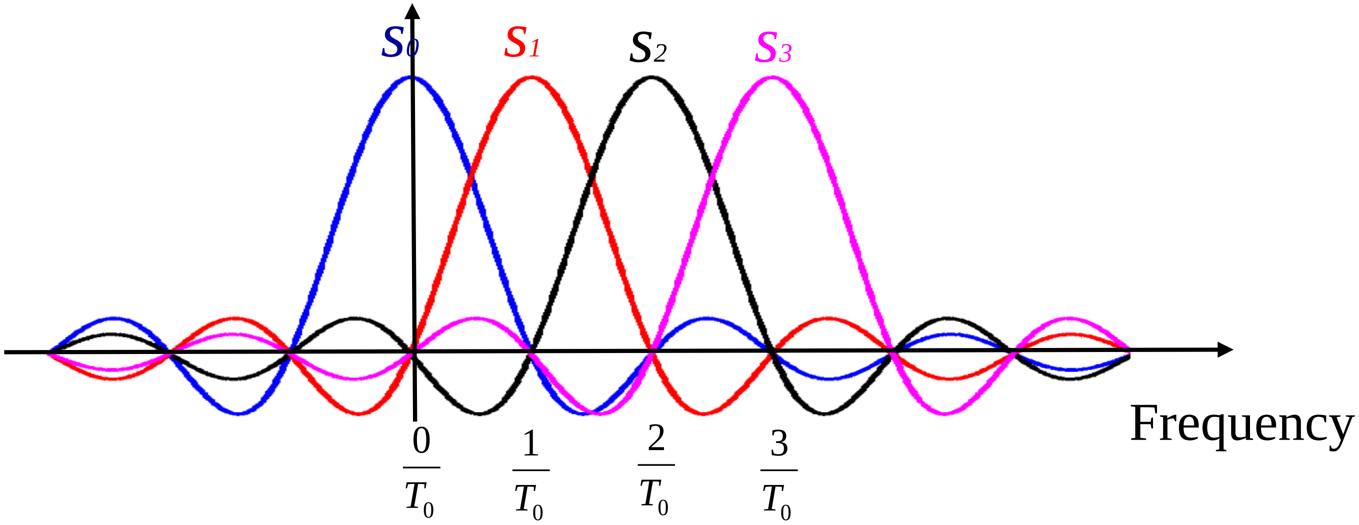
- **Example**
  - Consider a simple two-tap discrete-time channel with impulse response

$$H(z) = 1 + 0.9z^{-1}$$

find the matrix representation in time and frequency domain of an OFDM system with N = 8.

MCM: OFDM

- **OFDM: frequency domain**
  - Orthogonal in frequency domain ➔ no inter-carrier interference (ICI) in static channel.



OUTLINE

- Introduction
- OFDM
- **Challenges in Multicarrier Systems**

CHALLENGES: PAPR

- **Peak-to-average power ratio (PAPR)**

$$PAPR = \frac{\max_t |x(t)|^2}{E[|x(t)|^2]}$$

- A low PAPR
  - → the time domain amplitude does not change dramatically
  - → we can use more efficient non-linear power amplifier

- A high PAPR
  - → there is large variation in the time domain amplitude
  - → a lot of information is carried in the amplitude
  - → we have to use linear power amplifier, which usually has low efficiency

CHALLENGES: PAPR

- **Example**
  - What is the PAPR of a sin wave?

CHALLENGE: PAPR

- **PAPR in OFDM**
  - The time domain samples are obtained from IDFT of modulated symbols

$$x[n]=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}X[i]e^{j2\pi ni/N}$$

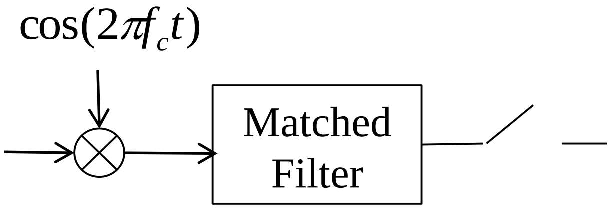
  - Based on central limit theorem, when N is large, then $x[n]$ follows zero-mean Gaussian distribution
  - The theoretical value of the PAPR is infinity (why?)
  - In practice, the PAPR increases approximately linearly with N

  - Large N → large PAPR ☹
  - Large N → small overhead due to cyclic prefix ☺

  - There are many ways to reduce PAPR in practical system
    - Clipping the peak signal
    - Special coding techniques
    - ……

CHALLENGE: FREQUENCY AND TIMING OFFSET

- **Frequency and timing offset**



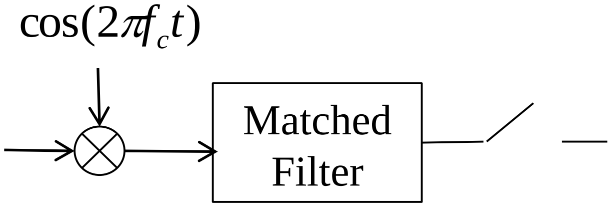
  - Frequency offset:
    - Due to the inaccuracy of the carrier frequency oscillator, there might be a small error in f_c
    - E.g. if the oscillator is accurate to 1 part per million, then the frequency offset for a 5GHz system is

$$\Delta f = f_c \cdot 10^{-6} =$$

    - The frequency offset will degrade the orthogonality among the subcarriers.

CHALLENGE: FREQUENCY AND TIMING OFFSET

- **Frequency and timing offset**



  - Timing offset
    - During the sampling of the time domain signal, there might be a small offset during the sampling
    - If the time offset is large, it will cause both ISI and ICI
    - If the time offset is small, it will cause ICI
    - Generally speaking, the ICI caused by the timing offset is smaller than that caused by frequency offset.

CASE STUDY: IEEE 802.11a

- **IEEE 802.11a: Wireless LAN standard**
  - Carrier frequency: 5 GHz
  - Total BW: 300 MHz
  - Channel BW: 20 MHz
  - Number of subcarriers: N = 64
    - Data: 48; zeros: 12; pilot: 4
  - Cyclic prefix: $\mu = 16$
  - Subcarrier spacing: 20 MHz/64 = 312.5 kHz
  - Sample period: Ts = 1/20 MHz = 50 ns
  - Maximum delay spread that can be handled by the system:
  - OFDM symbol time:
  - Code: convolutional code with rates 1/2, 2/3 or 3/4
  - Modulation: BPSK, QPSK, 16QAM, 64QAM
  - Minimum data rate:

  - Maximum data rate:

CASE STUDY: IEEE 802.11a

- **Example**
  - Find the data rate of an 802.11a system with 16QAM and 2/3 convolutional coding.