Department of Electrical Engineering
University of Arkansas

ELEG 5693 Wireless Communications
Math Review

Dr. Jingxian Wu
Jingxian.wu@sonoma.edu

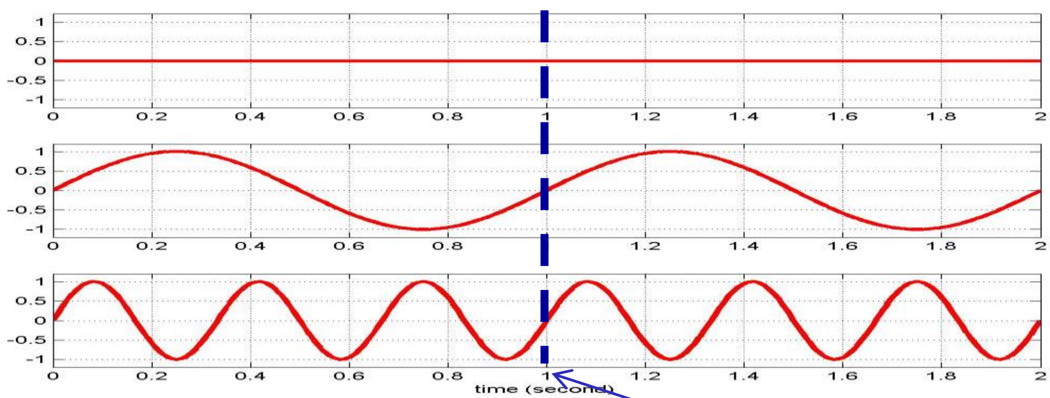
OUTLINE

- **Signals and System (Fourier analysis)**

- **Random variables and random process**

CONCEPT OF FREQUENCY

- **Frequency is the measurement of the number of times that a repeated event occurred in a unit time (1 second).**
  - Frequency measures how fast a signal can change within a unit period of time (or the measurement of the rate of change).
  - High frequency ➔ the signal changes fast
- **In signal processing, frequency is defined based on sinusoid signal → number of cycles per second.**
  - Each sinusoid signal is uniquely associated with a single frequency based on its rate of change.



f = 0 Hz

f = 1 Hz

f = 3 Hz

1 second

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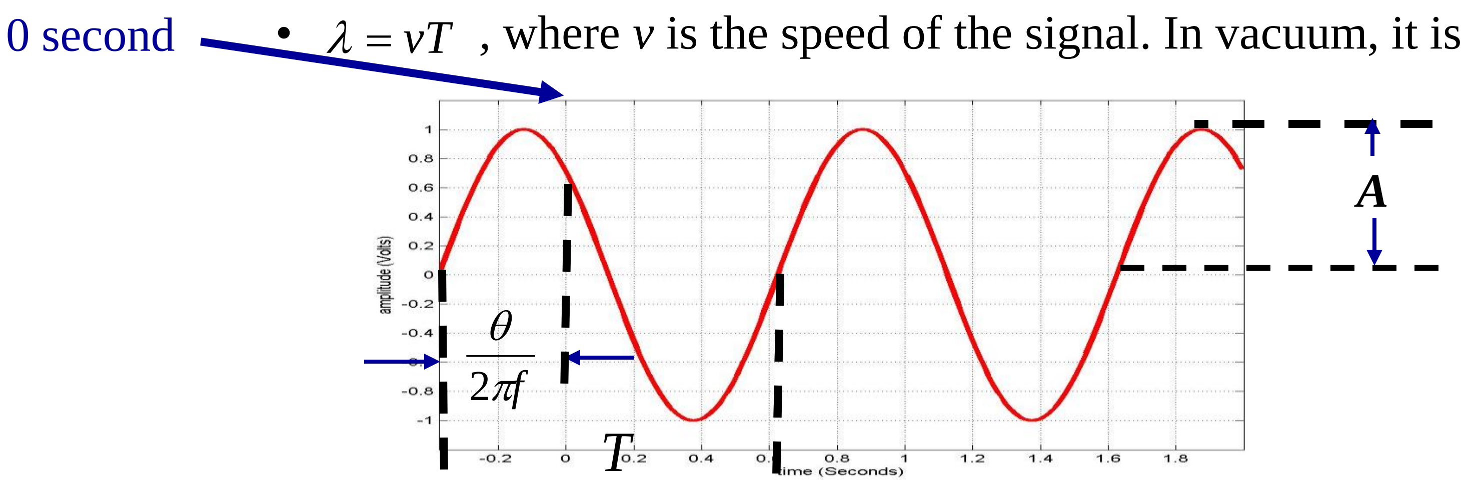
TIME DOMAIN SIGNAL

- **Sinusoid Signal**

$$s(t) = A\sin(2\pi f t + \theta)$$

  - f : frequency. in the unit of Hz (1/second)
  - A : amplitude. maximum strength in signal, usually in unit of volts
  - θ : phase. relative position in time
  - t : time. in the unit of second
  - T : period. Time for one cycle or one repetition, T=1/f.
  - λ : Wavelength. Distance of the signal waveform propagated in one period T.
    - $\lambda = vT$, where v is the speed of the signal. In vacuum, it is the speed of light.



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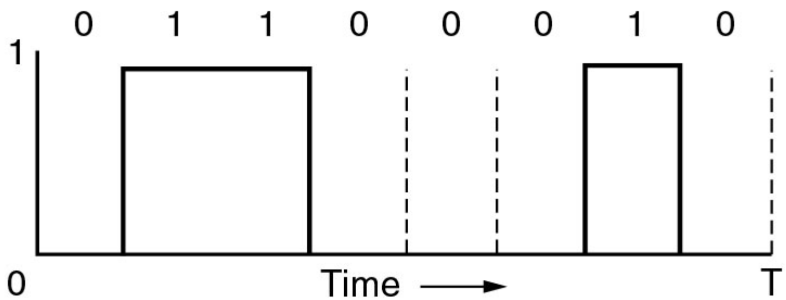
THEORY: FREQUENCY DOMAIN SIGNAL

- **Periodic signal can be written as the summation of a series of sinusoids (Fourier Series)**

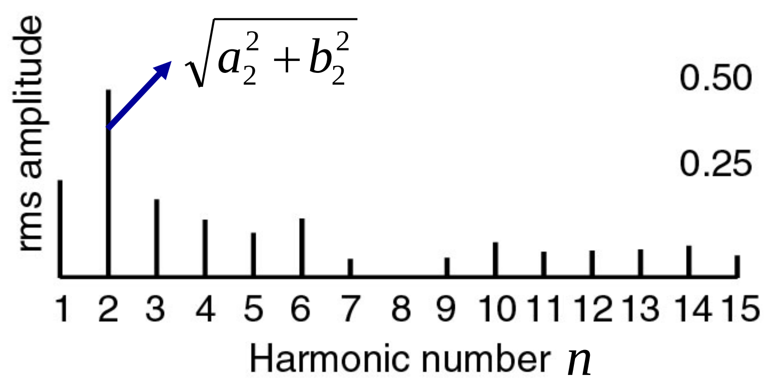
$$s(t) = \frac{1}{2}c + \sum_{n=1}^{\infty} a_n \sin(2\pi n f_n t) + \sum_{n=1}^{\infty} b_n \cos(2\pi n f_n t)$$

  - Each sinusoid component has a unique frequency
  - Each sinusoid component is called a “harmonic” of the original signal.
  - There is a one to one relationship between a signal and its Fourier series.
  - With Fourier series, we can represent the signal in the “Frequency Domain”



$s(t)$

Time Domain



$\sqrt{a_n^2 + b_n^2}$

Frequency Domain

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FOURIER TRANSFORM

Let *g*(*t*) be a non-periodic deterministic signal, expressed as function of *t*. Its frequency domain representation, *G*(*f*) , can be obtained from Fourier transform.

Fourier transform

$$G(f) = \int_{-\infty}^{+\infty} g(t)\exp(-j2\pi ft)dt$$

Inverse Fourier transform

$$g(t) = \int_{-\infty}^{+\infty} G(f)\exp(j2\pi ft)dt$$

$$j^2 = -1 \qquad \exp(jx) = \cos(x) + j\cdot\sin(x)$$

FOURIER TRANSFORM

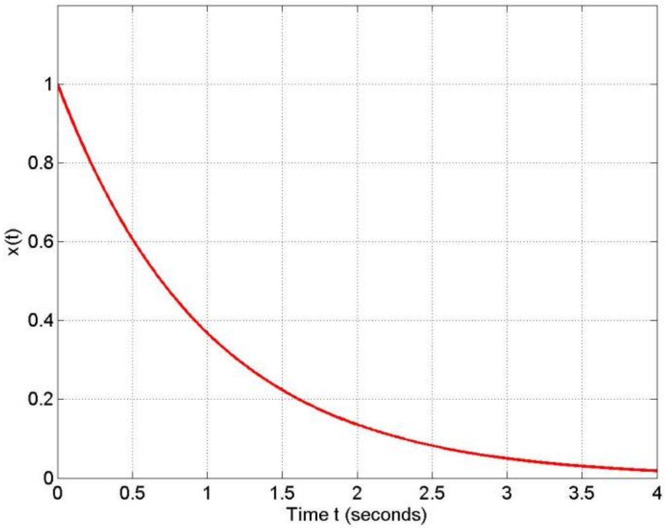
- **Example 1:**

Find the Fourier transform of $g(t) = \exp(-t)$, $t > 0$

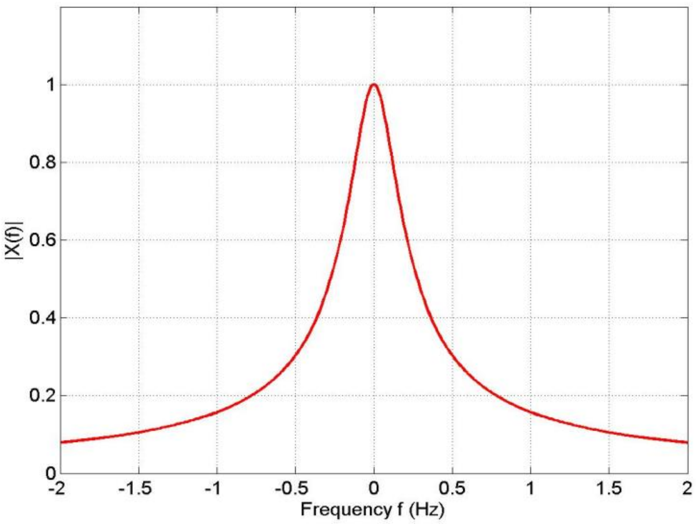
Sol.

FOURIER TRANSFORM

$$s(t) = e^{-t}, \quad t \geq 0 \xrightarrow{\text{Fourier Transform}} S(f) = \frac{1}{1 + j2\pi f}$$



Time Domain



Frequency Domain

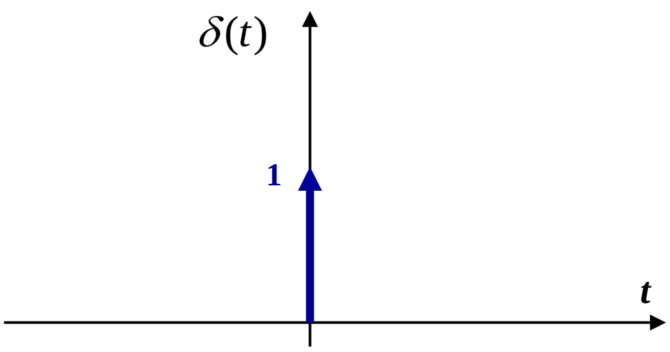
UNIT IMPULSE FUNCTION

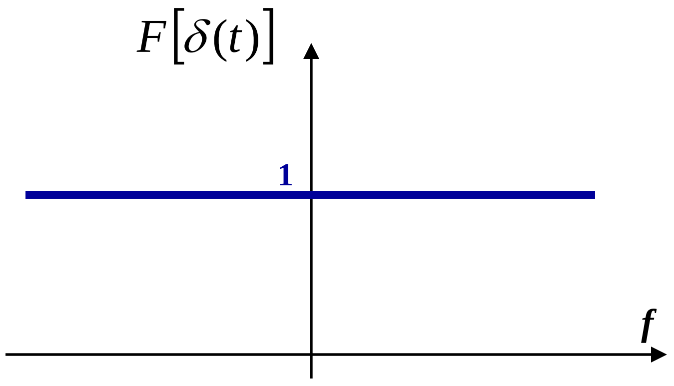
- **Unit impulse function (delta function)**

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1, \qquad \delta(0) = \infty, \qquad \delta(t) = 0, \forall t \neq 0$$

$$\delta(t)x(t) = \delta(t)x(0)$$

$$\int_{-\infty}^{+\infty} \delta(t)\exp(-j2\,ft)dt =$$





LINEAR TIME INVARIANT (LTI) SYSTEM

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- The output of the LTI system is the convolution of input *x*(t) with the channel impulse response *h*(t)

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- Time domain convolution ➔ frequency domain multiplication

$$X(f) = F\left[x(t)\right] \qquad H(f) = F\left[h(t)\right] \qquad Y(f) = F\left[y(t)\right]$$

$$Y(f) = X(f) \times H(f)$$

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LTI

- **Example:**

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$x(t) = \delta(t)$ $\quad$ $h(t) = \exp(-t), t \geq 0$ $\quad$ $y(t) = ?$

Method 1:

Method 2:

CORRELATION

- **The cross correlation of two deterministic, complex-valued function *a*(*t*) and *b*(*t*) is defined as**

$$R_{ab}(\tau) = \int_{-\infty}^{+\infty} a(t) b^*(t-\tau) dt$$

- **The auto-correlation of one deterministic, complex-valued function *a*(*t*) is defined as**

$$R_{a}(\tau) = \int_{-\infty}^{+\infty} a(t) a^*(t-\tau) dt$$

ENERGY

- **The energy of a signal s(t) is defined as**

$$E_s = \int_{-\infty}^{+\infty} |s(t)|^2 \, dt$$

  - If a signal has finite energy, the signal is called energy signal.
  - E.g. $s(t) = \exp(-t), t \geq 0$

$$E_s =$$

- **Relationship between energy and auto-correlation function**

$$E_s = \int_{-\infty}^{+\infty} |s(t)|^2 \, dt = \int_{-\infty}^{+\infty} s(t) s^*(t-0) dt = R_s(0)$$

POWER

- **The power of a signal s(t) is defined as**

$$P_s = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |s(t)|^2 \, dt$$

  - If a signal has finite power, the signal is called power signal
  - For periodic signal with period T

$$P_s = \frac{1}{T} \int_0^T |s(t)|^2 \, dt$$

  - E.g. $s(t) = A\sin(2\pi f t + \theta)$

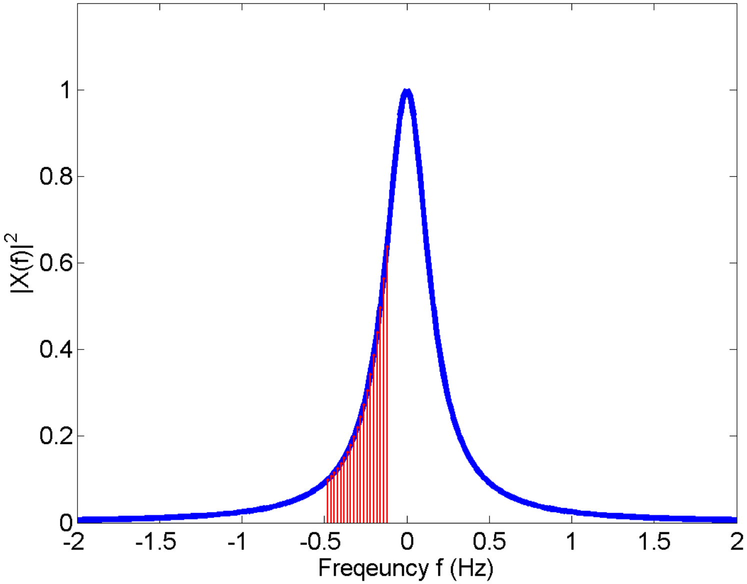
$$P_s =$$

PARSEVAL'S THEOREM & ESD

$$E_s = \int_{-\infty}^{+\infty} |s(t)|^2 dt = \int_{-\infty}^{+\infty} |S(f)|^2 df$$

- $|S(f)|^2$ **is called the energy spectrum density (ESD) of the signal**
  - ESD represents the energy distribution in the frequency domain.

e.g. ESD of $s(t) = \exp(-t), t \geq 0$ $\qquad \dfrac{1}{|1+j2\pi f|^2}$



$\int_{f_1}^{f_2} |S(f)|^2 df$:The signal energy in the frequency range of $[f_1, f_2]$

OUTLINE

- **Linear system theory (Fourier analysis) (Appendix A)**
- **Random variables and random process (Appendix C)**

DISCRETE RANDOM VARIABLES

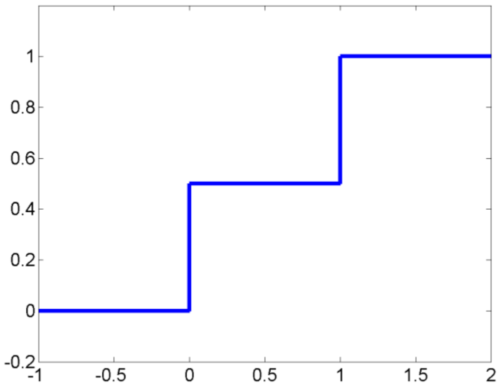
- **Example 1: coin toss**
  - Define random variable (RV) X.
    - X = 0: coin head
    - X = 1: coin tail
  - Probability Mass Function (PMF):
    - P(X = 0) = 0.5, P(X = 1) = 0.5
- **Example 2: pick 1 ball from 10 balls (2 black, 3 white, 5 red)**
  - Define RV X
    - X = 0: a black ball is picked
    - X = 1: a white ball is picked
    - X = 2: a red ball is picked
  - PMF:

CUMLATIVE DISTRIBUTION FUNCTION (CDF)

$$F_X(x) = \mathrm{P}(X \le x)$$

- **Example 1: toss coin**

$$F_X(x) = \begin{cases} 0, & x < 0 \\ P(X = 0) = 0.5, & 0 \le x < 1 \\ P(X = 0) + P(X = 1) = 1, & x \ge 1 \end{cases}$$



- **Example 2: pick 1 balls from 10 balls**

$$F_X(x) = \begin{cases} & x < 0 \\ & 0 \le x < 1 \\ & 1 \le x < 2 \\ & x \ge 2 \end{cases}$$

CONTINUOUS RV

- **RV X can take continuous values**
  - E.g.: X represents the average temperature
- **CDF**

$$\boxed{F_X(x) = \mathrm{P}(X \le x)}$$

$$F_X(-\infty) = 0 \qquad F_X(\infty) = 1$$

- **Probability density function (pdf)**

$$\boxed{f_X(x) = \frac{dF_X(x)}{dx}}$$

Interpretation: for small Δx ,

$$f_X(x) = \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x} = \frac{\mathrm{P}(x < X < x+\Delta x)}{\Delta x}$$

$$f_X(x)\Delta x = \mathrm{P}(x < X < x+\Delta x)$$

or, the probability that the RV X is in the range of $(x, x+\Delta x)$

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PDF AND CDF

- Relationship between pdf and CDF

$$f_X(x) = \frac{dF_X(x)}{dx} \qquad F_X(x) = \int_{-\infty}^{x} f_X(y)dy$$

$$\mathrm{P}(x_2 < X < x_1) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(y)dy$$

$$\mathrm{P}(X < \infty) = F_X(\infty) = \int_{-\infty}^{+\infty} f_X(y)dy = 1$$

CDF AND PDF

- **Example:**

The pdf of uniform distribution is: $f_X(x) = \dfrac{1}{b-a}, \quad a < x < b$

The corresponding CDF is:

Let $a = 0$, $b = 1$, the probability that X is in the range of [0.5, 0.6] is:

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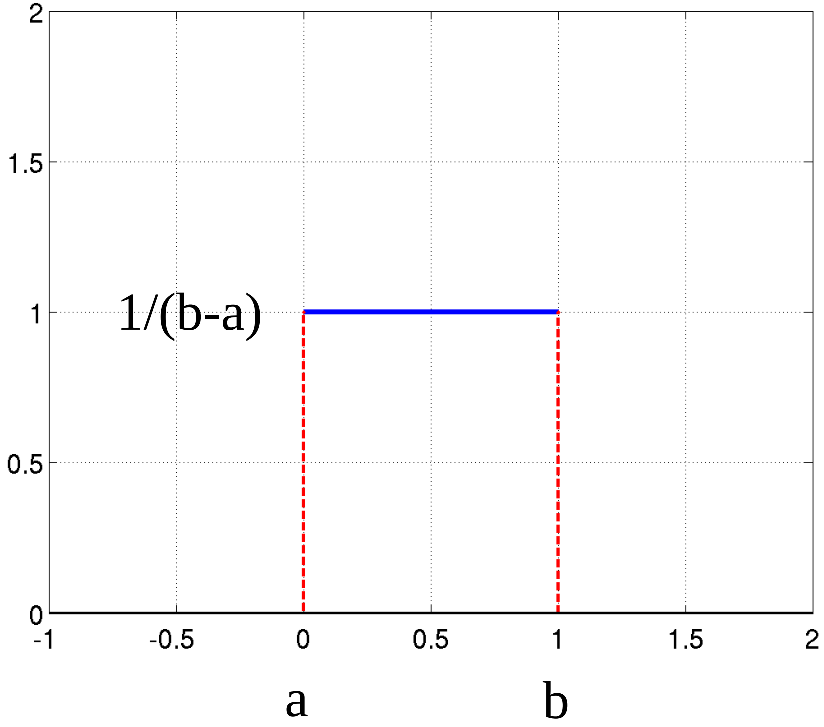
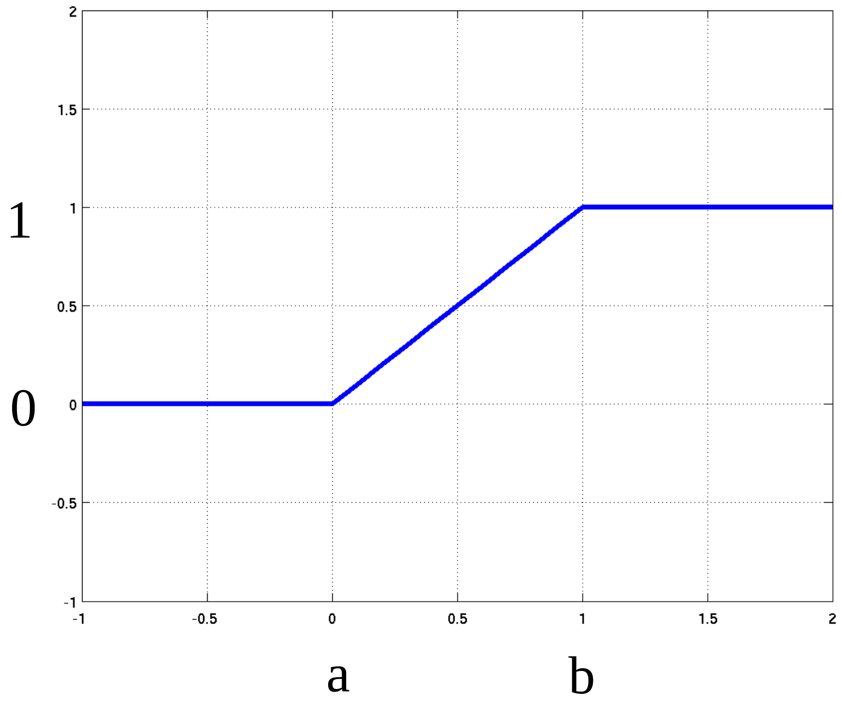
CDF AND PDF



pdf of uniform distribution



cdf of uniform distribution

EXPECTATION (MEAN)

- **Discrete-time RV**

$$m_X = E(X) = \sum_i x_i P(X = x_i)$$

  – Weighted sum

- **Continuous-time RV**

$$m_X = E(X) = \int_{-\infty}^{+\infty} y f_X(y) dy$$

- **Example: find the expectation (mean) of the uniform distribution**

$$f_X(x) = \frac{1}{b-a}, \quad a < x < b$$

VARIANCE

- **Discrete RV**

$$\sigma^2 = \mathrm{E}\left[(X - m_X)^2\right] = \sum_i \left(x_i - m_X\right)^2 P(X = x_i)$$

- **Continuous RV**

$$\sigma^2 = \mathrm{E}\left[(X - m_X)^2\right] = \int_{-\infty}^{+\infty} \left(y - m_X\right)^2 f_X(y)dy$$

- $$\boxed{\sigma^2 = \mathrm{E}\left(X^2\right) - m_X^2}$$

  – Proof:

VARIANCE

- **Example: find the variance of the uniform distribution**

$$f_X(x) = \frac{1}{b-a}, \quad a < x < b$$

  - Sol:

GAUSSIAN DISTRIBUTION

- **The pdf of Gaussian distribution is**

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$

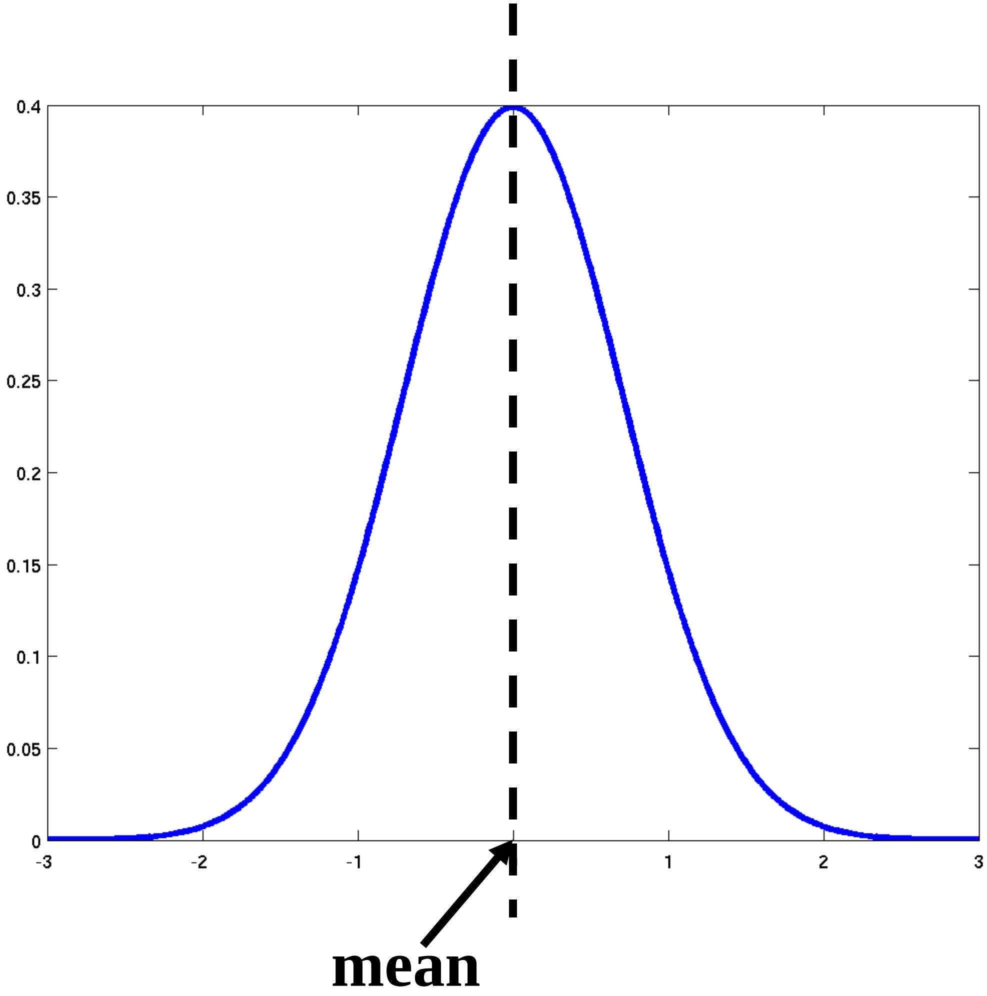
  - The pdf is determined by two parameters:

$$m = E(X) \qquad \text{mean}$$

$$\sigma = \sqrt{E(X - m_X)^2} \qquad \text{standard deviation}$$

- **The sum of Gaussian RVs is still Gaussian distributed**

GAUSSIAN DISTRIBUTION



JOINT DISTRIBUTION

- **Consider two RVs, *X*, and, *Y*. The joint CDF of X and Y is**

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

- **Joint pdf**

$$f_{X,Y}(x,y) = \frac{\partial F_{X,Y}(x,y)}{\partial x \partial y}$$

- **Given the joint pdf, we can find the marginal pdf of each RV**

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy \qquad f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx$$

JOINT DISTRIBUTION

- **Example:**

$$f_{X,Y}(x,y)=\begin{cases} x+y, & 0\le x\le 1, 0\le y\le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x)=?,\ f_Y(y)=?$$

  – Sol:

- **Independent**
  – If $f_{X,Y}(x,y)=f_X(x)f_Y(y)$, then X and Y are independent.

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CORRELATION

- **The correlation of two RVs *X* and *Y* is calculated as**

$$E\left[XY\right]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xyf_{XY}(x,y)dxdy$$

- **The covariance of two RVs *X* and *Y* is**

$$E\left[\left(X-m_X\right)\left(Y-m_Y\right)\right]=E[x]E[Y]-m_X m_Y$$

- **If** $E\left[XY\right]=E(X)E(Y)$ **, then X and Y are uncorrelated.**

If X and Y are independent → they must be uncorrelated.

Proof:

CORRELATION

- **Example:**

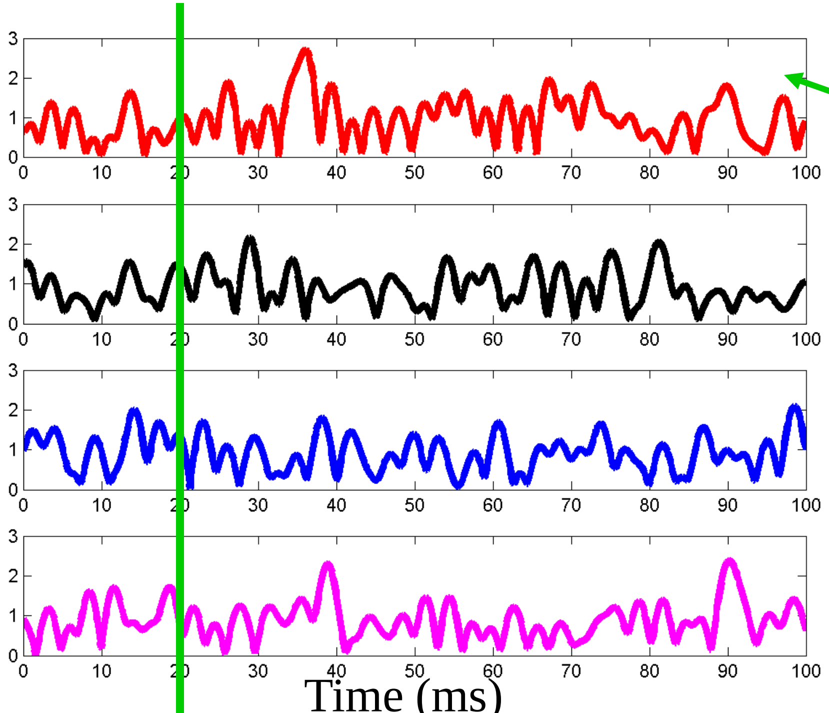
$$f_{X,Y}(x,y)=\begin{cases} x+y, & 0\le x\le 1, 0\le y\le 1 \\ 0, & \text{otherwise} \end{cases}$$

Are X, Y uncorrelated?

– Sol:

RANDOM PROCESS

- **Random process $X(t)$: an RV changes w.r.t. time**
  - $X(t)$ is a function of time t
  - At any time instant t_0, $X(t_0)$ is a random variable.



Each realization is called a sample function of the random process

At each time instant, we have a RV

mean: (ensemble average)

$$m_X(t_0) = E[X(t_0)]$$

RANDOM PROCESS

- Let $X(t)$ be a random process,
  at time t_1, we have a RV $X(t_1)$;
  at time t_2, we have another RV $X(t_2)$.
  Then we have the joint distribution functions

$$F_{X(t_1)X(t_2)}(x_1, x_2) = P\left[X(t_1) < x_1, X\left(t_2\right) < x_2\right]$$

$$f_{X(t_1)X(t_2)}(x_1, x_2) = \frac{\partial F_{X(t_1)X(t_2)}(x_1, x_2)}{\partial x_1 \partial x_2}$$

AUTO-CORRECTION FUNCTION (ACF)

- Let $X(t)$ be a random process,

  at time t_1 , we have a RV $X(t_1)$;

  at time t_2 , we have another RV $X(t_2)$.

  Then we can calculate the correlation of $X(t_1)$ and $X(t_2)$:

$$R_X(t_1, t_2) = E\left[X(t_1)X^*(t_2)\right]$$

$R_X(t_1, t_2)$ **is defined as the autocorrelation function of *X*(*t*)**

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WIDE-SENSE STATIONARY (WSS)

- **A random process is wide-sense stationary (WSS) if the following two conditions are satisfied**

1. $$m_X(t) = E\left[X(t)\right] = m_x$$

2. $$R_X(t_1, t_2) = R_X(t_1 + h, t_2 + h) = R_X(t_1 - t_2)$$

– The first condition states that the mean of the random process is independent of time.

– The second condition states that the auto-correlation function is only dependent on the time difference between the two RVs, and it's independent of the starting time.

$$E[X(t_1)X(t_2)] = E\left[X(t_1 + h)X(t_2 + h)\right]$$

- The autocorrelation function of WSS process is usually represented as

$$R_X(\tau)$$

POWER SPECTRUM DENSITY

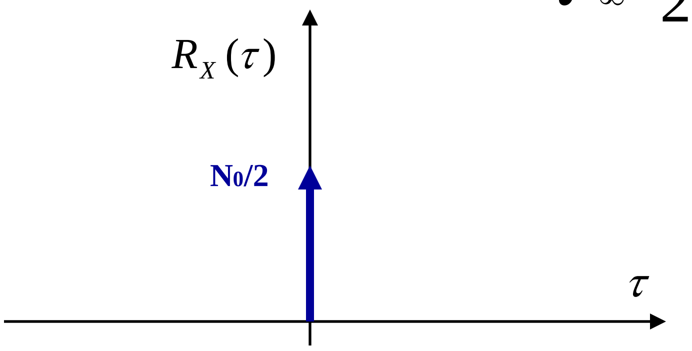
- The power spectrum density (PSD) of WSS random process is defined as the Fourier transform of the auto-correlation function of the process

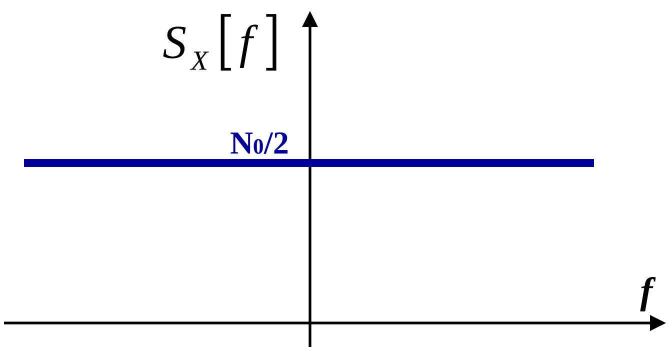
$$S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau$$

- Power spectrum density represents the power distribution in the frequency domain.
- E.g.

$$R_x(\tau) = \frac{N_0}{2}\delta(\tau)$$

$$S_x(f) = \int_{-\infty}^{+\infty} \frac{N_0}{2}\delta(\tau) d\tau = \frac{N_0}{2}$$





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RANDOM PROCESS PASS THROUGH LTI

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- **If x(t) is a WSS random process, then y(t) is a WSS random process as well.**

The relationship between mean of y(t) and mean of x(t)

$$E[y(t)] = E\left[\int_{-\infty}^{+\infty} h(t-\tau)x(\tau)d\tau\right] = \int_{-\infty}^{+\infty} h(t-\tau)E[x(\tau)]d\tau$$

The relationship between PSD of y(t) and PSD of x(t)

$$S_Y(f) = |H(f)|^2 S_X(f)$$