Department of Electrical Engineering
University of Arkansas

ELEG5663 Communication Theory
Ch. 3 Baseband Demodulation and Detection

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OUTLINE

- **Signals and noise**
- **Detection of binary signals in AWGN**
- **Matched Filter**
- **Intersymbol Interference (ISI)**
- **Equalization**

SIGNAL AND NOISE

- **Baseband demodulation and detection**
  - Recover the received baseband signal from distortions caused by noise and intersymbol interference (ISI)
- **Equivalence theorem**
  - The following two operations are equivalent
    - 1. performing bandpass linear signal processing; 2. converting the processed signal to baseband.
    - 1. converting the received signal to baseband; 2. performing baseband linear signal process.
  - It's usually more expensive to perform bandpass linear signal processing.
  - Baseband demodulation/detection can be used for both bandpass system and baseband system.
  - Simulation of communication system is usually performed in baseband only
    - Faster simulation
    - Yields the same result

SIGNAL AND NOISE

- **Binary Communication System Model**
  - The transmitted signal over a symbol interval (0, T) is represented by

$$s_i(t) = \begin{cases} s_1(t), & 0 \le t \le T, \quad '1' \\ s_2(t), & 0 \le t \le T, \quad '0' \end{cases}$$

  - The received signal through LTI channel

$$r(t) = s_i(t) \otimes h_c(t) + n(t)$$

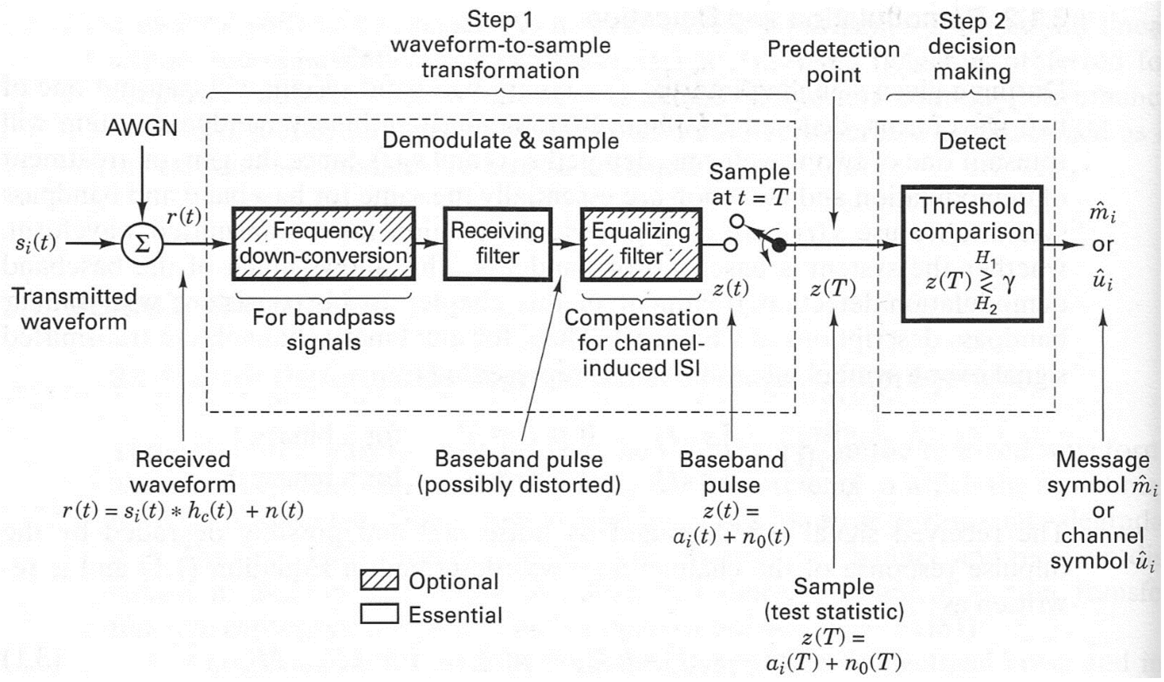
  - For ideal distortionless channel, $h_c(t) = \delta(t)$

$$r(t) = s_i(t) + n(t)$$

    - AWGN channel

SIGNAL AND NOISE

- **Receiver structure**
  - Demodulation (frequency down-conversion, receiving filter)
    - Recovers a waveform to an undistorted baseband signal
  - Equalization
    - Mitigates the effects caused by intersymbol interference (ISI).
  - Detection
    - Based on the demodulated signal, make decision of ‘1’ or ‘0’.



SIGNAL AND NOISE: VECTORIAL REPRESENTATION

- **N-dimensional Orthogonal Space**
  - Basis functions: $\psi_j(t) \qquad j = 1, 2, \cdots, N$
  - Orthogonal functions

$$\int_0^T \psi_j(t)\psi_k(t)dt = K_j \delta_{jk}$$

$$\delta_{jk} = \begin{cases} 1, & j = k \\ 0, & otherwise \end{cases}$$

  - If $K_j = 1$: orthonormal
  - Any arbitrary finite set of waveforms $\{s_j(t)\} \qquad j = 1, \cdots, M$ can be expressed a linear combination of N orthogonal waveforms

$$\begin{aligned} s_1(t) &= a_{11}\psi_1(t) + a_{12}\psi_2(t) + \cdots + a_{1N}\psi_N(t) \\ s_2(t) &= a_{21}\psi_1(t) + a_{22}\psi_2(t) + \cdots + a_{2N}\psi_N(t) \\ &\vdots \\ s_M(t) &= a_{M1}\psi_1(t) + a_{M2}\psi_2(t) + \cdots + a_{MN}\psi_N(t) \end{aligned}$$

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\psi_j(t)$$

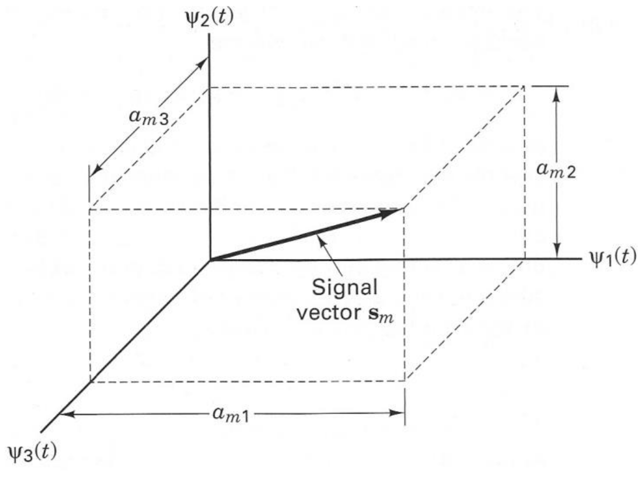
$$a_{ij} = \frac{1}{K_j}\int_0^T s_i(t)\psi_j(t)dt$$

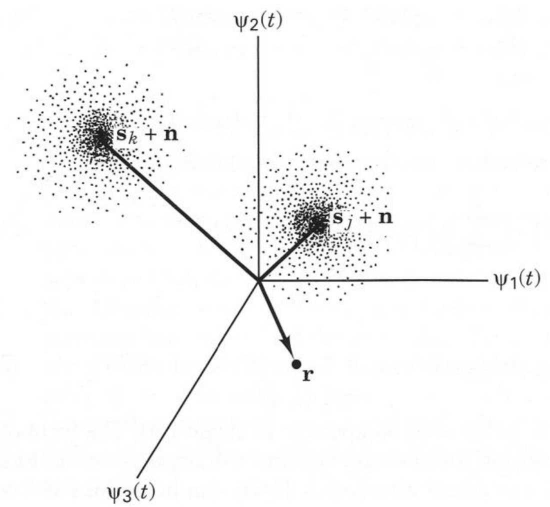
SIGNAL AND NOISE: VECTORIAL RREPSENTATION

- **Vectorial signal representation**
  - An example of orthogonal space: Fourier series

$$\psi_k(t) = e^{j\frac{2\pi n}{T}t} \qquad s(t) = \sum_n a_n e^{j\frac{2\pi n}{T}t}$$

  - Once the orthogonal space is chosen, there is a one-to-one mapping between $s_i(t)$ and $\mathbf{s}_i = (a_{i1}, a_{i2}, \cdots, a_{iN})$
    - The signal can be equivalently represented by the vector $\mathbf{s}_i = (a_{i1}, a_{i2}, \cdots, a_{iN})$
  - Similarly, the noise, $n(t)$ can also be represented by the vector $\mathbf{n} = (n_1, n_2, \cdots, n_N)$
  - The received signal, $r(t)$ can be represented by $\mathbf{r} = \mathbf{s}_i + \mathbf{n}$





SIGNAL AND NOISE: VECTORIAL REPRESENTATION

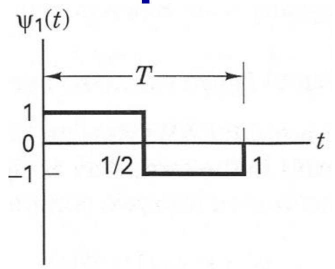
- **Waveform energy**
  - The energy of the signal waveform $s_i(t)$ over a symbol interval T

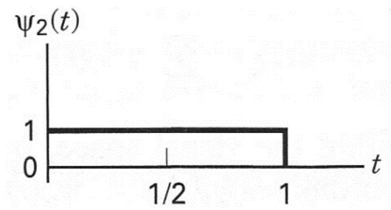
$$E_i = \int_0^T s_i^2(t)dt = \sum_{j=1}^{N} a_{ij}^2 K_j$$

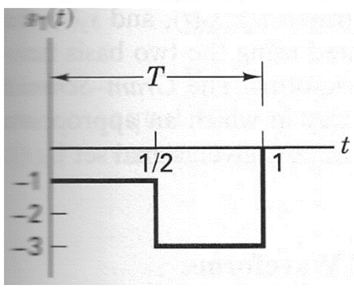
  - Proof
  - Generalized Parsavel's theorem.

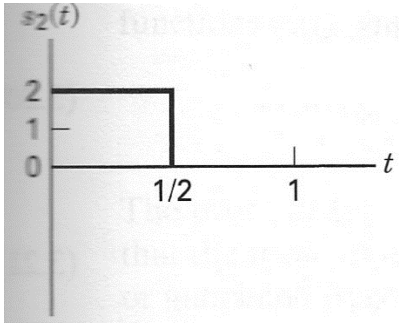
SIGNAL AND NOISE: VECTORIAL REPRESENTATION

- **Example**









SIGNAL AND NOISE: VECTORIAL REPRESENTATION

- **Representation of white noise with orthogonal waveforms**
  - The white noise can be represented as

$$n(t) = \sum_{j=1}^{N} n_j \psi_j(t) + \tilde{n}(t) \qquad n(t) \Leftrightarrow \mathbf{n} = (n_1, n_2, \cdots, n_N)$$

  - $\tilde{n}(t)$ doesn't interfere with signal (we don't need to consider it)
  - $n(t)$ has unlimited power (why?)
    - But $n(t) - \tilde{n}(t)$ has limited power.

- **Variance (power) of white noise per dimension**
  - If the two-sided PSD of white noise is $\frac{N_0}{2}$

$$\sigma^2 = E\left\{ \left[\int_0^T n(t)\psi_j(t)dt \right]^2 \right\} =$$

SIGNAL AND NOISE: SNR

- E_b / N_0
  - Normalized signal to noise ratio (SNR). The figure of merit in digital communication system.
    - E_b: energy of 1 bit
    - N_0: single-sided PSD of noise

SNR

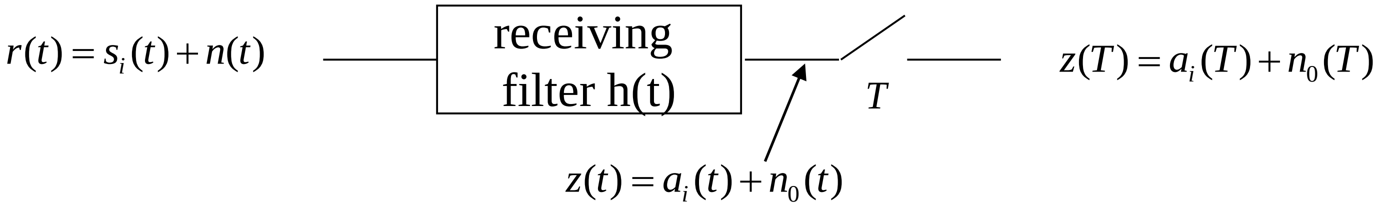
$$\frac{E_b}{N_0} = \frac{ST_b}{N/W} = \frac{S}{N} \cdot \frac{W}{R}$$

  - It allows the fair comparison between systems with different modulation levels
    - E.g. a binary modulated system and a 8-ary modulated system

OUTLINE

- Signals and noise
- **Detection of binary signals in AWGN**
- Matched Filter
- Intersymbol Interference (ISI)
- Equalization

DETECTION: MAXIMUM LIKELIHOOD DETECTION



- **After receiving filter h(t):**

$$a_i(t) = s_i(t) \otimes h(t) \qquad n_0(t) = n(t) \otimes h(t)$$

$$z(t) = a_i(t) + n_0(t)$$

- **After sampler**

$$z(T) = a_i(T) + n_0(T)$$

  – Or:

$$z = a_i + n_0$$

  – the noise sample n_0 is obtained from linear transformation of AWGN
    - n_0 is zero mean Gaussian distributed with variance σ_0^2

$$p_{n_0}(x) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma_0^2} \right]$$

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DETECTION: MAXIMUM LIKELIHOOD DETECTION

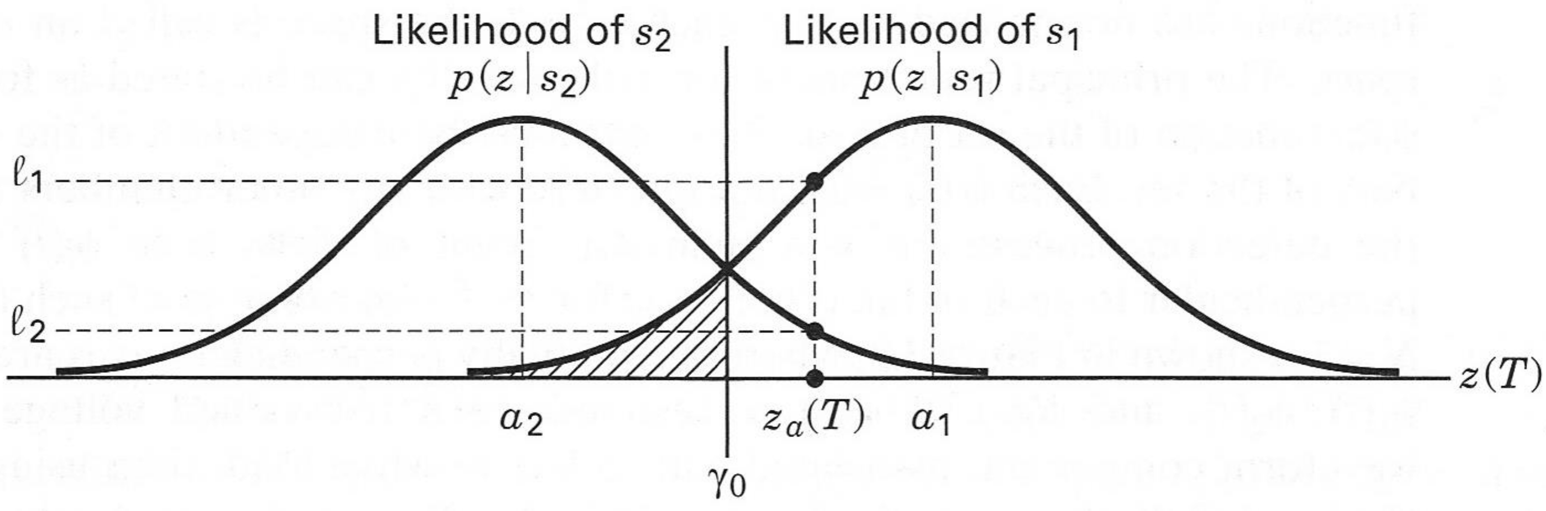
- **Likelihood function**

$$z = a_i + n_0$$

  - n_0: linear transformation of Gaussian. $n_0 \sim N(0, \sigma_0^2)$
  - When a_i is transmitted, $z \mid a_i \sim N(a_i, \sigma_0^2)$

$$p(z \mid s_i) = p(z \mid a_i) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{z - a_i}{\sigma_0} \right)^2 \right]$$

likelihood function



DETECTION: MAXIMUM LIKELIHOOD DETECTION

- **Maximum likelihood receiver**
  - Decision rule: for a certain threshold γ_0
    - When $z > \gamma_0$, make decision that '1' is transmitted.
    - When $z < \gamma_0$, make decision that '2' is transmitted.
    - How do we choose γ_0 ? ➔ minimize error probability.
  - The error probability when '1' is transmitted.

$$P(E \mid s_1) = P(\text{choose '2'} \mid s_1) = P(z < \gamma_0 \mid s_1)$$

    - When s_1 is transmitted, an error will occur if $z < \gamma_0$

$$P(E \mid s_1) = P(z < \gamma_0 \mid s_1) = \int_{-\infty}^{\gamma_0} p(z \mid s_1) dz$$

  - The error probability when '2' is transmitted.
    - When s_2 is transmitted, an error will occur if $z > \gamma_0$

$$P(E \mid s_2) = P(z > \gamma_0 \mid s_2) = \int_{\gamma_0}^{+\infty} p(z \mid s_2) dz$$

  - Error probability:

$$P(E) = P(E \mid s_2) P(s_2) + P(E \mid s_1) P(s_1)$$

$$= P(s_1) \int_{-\infty}^{\gamma_0} p(z \mid s_1) dz + P(s_2) \int_{\gamma_0}^{+\infty} p(z \mid s_2) dz$$

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DETECTION: MAXIMUM LIKELIHOOD DETECTION

- **Maximum likelihood receiver (Cont'd)**
  - Minimize $P(E)$
    - Differentiate $P(E)$ with respect to γ_0

$$\frac{\partial P(E)}{\partial \gamma_0} = p(\gamma_0 \mid s_1)P(s_1) - p(\gamma_0 \mid s_2)P(s_2)$$

    - Optimum γ_0 ➔ $\frac{\partial P(E)}{\partial \gamma_0} = 0$
      - The optimum threshold should satisfy the following condition

$$\frac{p(\gamma_0 \mid s_1)}{p(\gamma_0 \mid s_2)} = \frac{P(s_2)}{P(s_1)}$$

DETECTION: MAXIMUM LIKELIHOOD RECEIVER

- **Maximum likelihood receiver: threshold in the presence of AWGN**

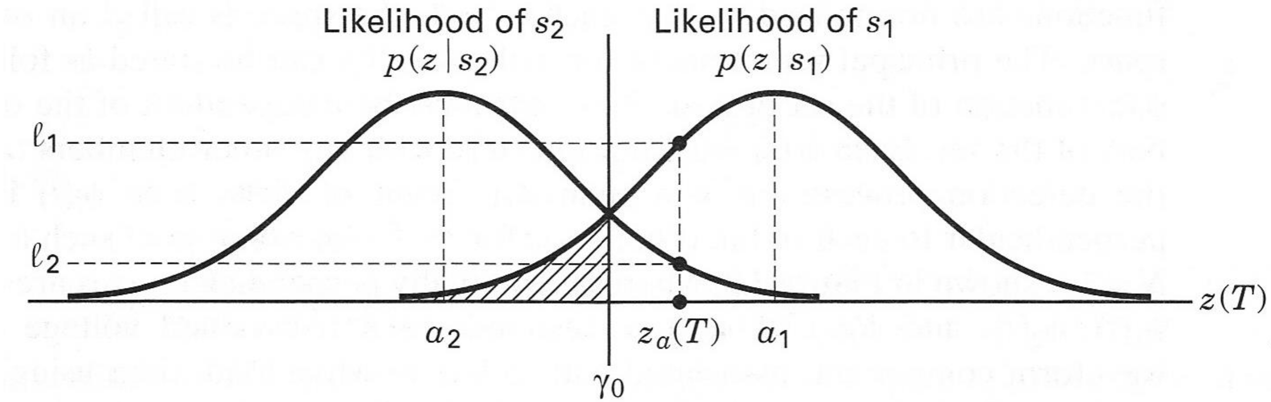
$$z = a_i + n_0$$

  - n_0: linear transformation of Gaussian. $n_0 \sim N(0, \sigma_0^2)$
  - When a_i is transmitted, $z \mid a_i \sim N(a_i, \sigma_0^2)$

$$p(z \mid s_i) = p(z \mid a_i) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z - a_i}{\sigma_0}\right)^2 \right]$$ likelihood function

  - the threshold γ_0 when $P(s_1) = P(s_2)$

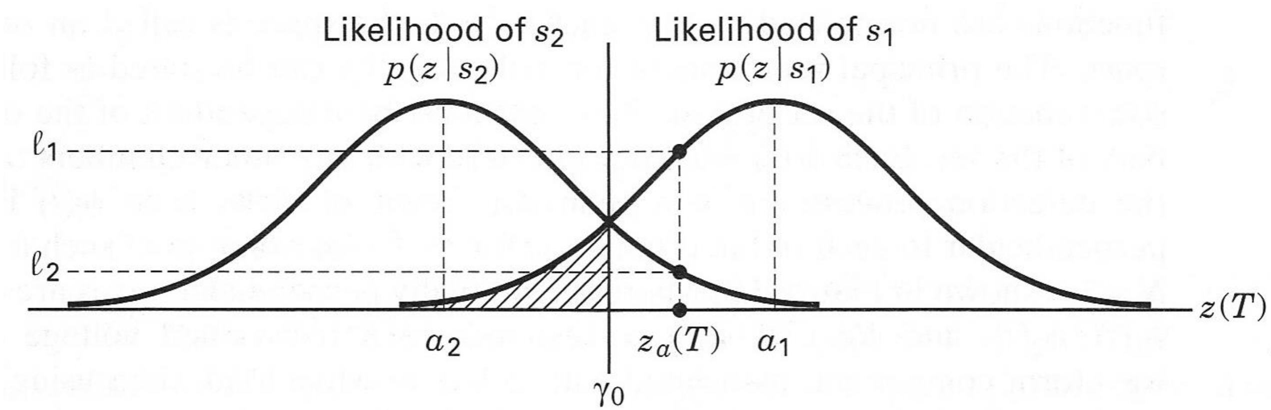
$$\gamma_0 = \frac{a_1 + a_2}{2}$$



DETECTION: MAXIMUM LIKELIHOOD RECEIVER

- **Maximum likelihood receiver: threshold in the presence of AWGN**
  - The decision rule that minimize $P(E)$ when $P(s_1) = P(s_2)$

    $\frac{p(z|s_1)}{p(z|s_2)} > 1$ ➔ '1' $\frac{p(z|s_1)}{p(z|s_2)} < 1$ ➔ '2'
  - Likelihood function: $p(z|s_1)$ $p(z|s_2)$
  - Likelihood ratio: $\frac{p(z|s_1)}{p(z|s_2)}$



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DETECTION: MAXIMUM LIKELIHOOD RECEIVER

- **Error probability**
  - Definition: Gaussian Q function:

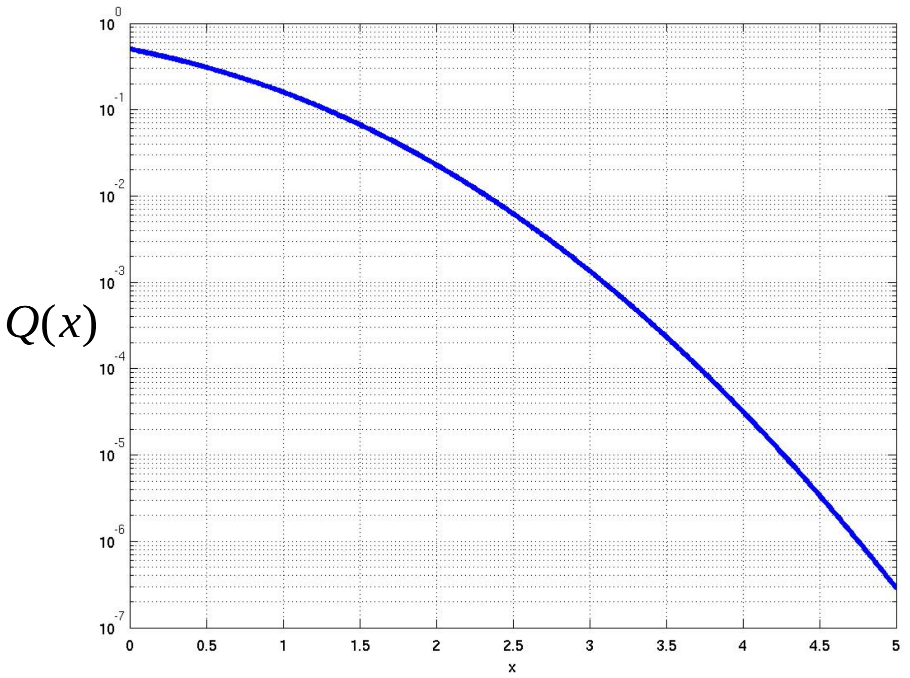
$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{u^2}{2}\right) du$$

  - Error probability:

$$P(E \mid s_1) = \int_{-\infty}^{\gamma_0} p(z \mid s_1) dz =$$

$$P(E \mid s_2) = \int_{\gamma_0}^{+\infty} p(z \mid s_2) dz =$$

$$P(E) = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$$



DETECTION: MAXIMUM LIKELIHOOD RECEIVER

- **Example**
  - Assume that in a binary digital communication system, the signal component after receiving filter and sampling is $a_1 = 3$V when '1' is transmitted, it is $a_2 = 1$V when '0' is transmitted. If the Gaussian noise at the output of receiving filter has unit variance. Assume '1' and '0' have equal probability
    - 1. Find the detection threshold.
    - 2. Find the error probability.
    - 3. If the output of the receiving filter and sampler is $z = 2.4$V, what should we detect?
    - 4. Find the values of $p(z | s_1)$ and $p(z | s_2)$
    - 5. Write the equation that can be used to solve the threshold when P('1') = 0.7, and P('0') = 0.3. Solve the threshold

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DETECTION: MATCHED FILTER

$$P(E) = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$$

- **Error probability depends on**
  - 1. $a_1 - a_2$

    $$a_i = \int_{-\infty}^{+\infty} s_i(T-\tau)h(\tau)d\tau$$

    $$a_1 - a_2 = \int_{-\infty}^{+\infty} [s_1(T-\tau) - s_2(T-\tau)]h(\tau)d\tau = \int_{-\infty}^{+\infty} s_0(T-\tau)h(\tau)d\tau$$

    - $a_1 - a_2$ depends on: $s_0(t) = s_1(t) - s_2(t)$ and $h(t)$
  - 2. σ_0

    $$n_0(t) = \int_{-\infty}^{+\infty} n(T-\tau)h(\tau)d\tau$$

    - σ_0 depends on: $n(t)$ and $h(t)$
  - We want to design receiving filter $h(t)$ to maximize the following SNR:

$$\frac{(a_1 - a_2)^2}{\sigma_0^2} = \frac{\left|\int_{-\infty}^{+\infty} s_0(T-\tau)h(\tau)d\tau\right|^2}{\sigma_0^2}$$

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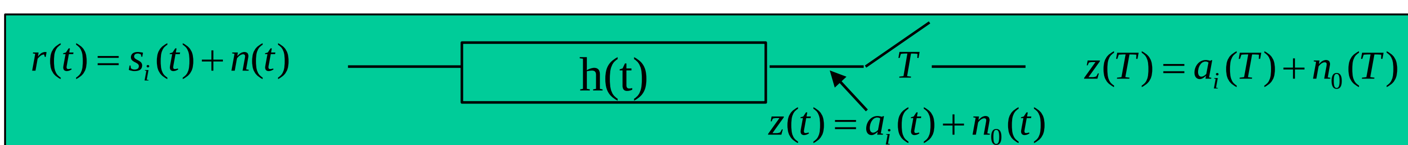


DETECTION: MATCHED FILTER

- **Matched filter: Design**
  - Signal:

inverse Fourier transform

$$A_0(f) = S_0(f)H(f) \Rightarrow a_0(t) = \int_{-\infty}^{+\infty} S_0(f)H(f)e^{j2\pi ft}df$$

$$[a_0(T)]^2 = \left|\int_{-\infty}^{+\infty} S_0(f)H(f)e^{j2\pi fT}df\right|^2$$

  - noise:

$$\psi_n(f) = \frac{N_0}{2}|H(f)|^2 \Rightarrow \sigma_0^2 = \frac{N_0}{2}\int_{-\infty}^{+\infty}|H(f)|^2\,df$$

  - SNR maximization

Cauchy-Schwartz Inequality

$$\left|\int f(x)g(x)dx\right|^2 \le \int |f(x)|^2 dx \int |g(x)|^2 dx$$

  - Equality holds if $g(x) = kf^*(x)$

DETECTION: MATCHED FILTER

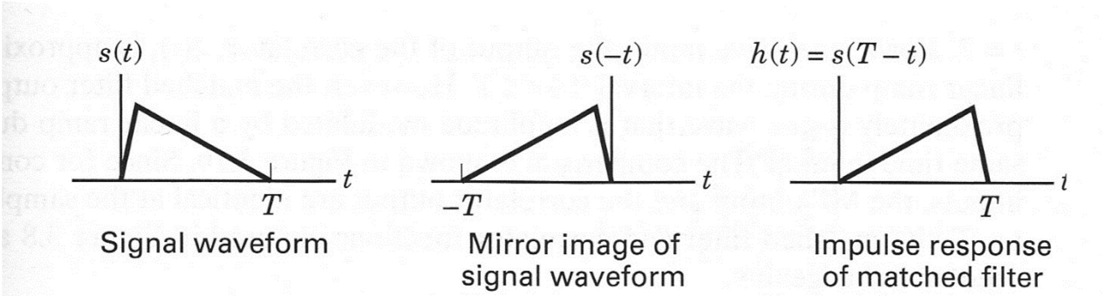
- **Matched filter**
  - The impulse response of matched filter is

$$h(t) = ks_0^*(T-t)$$

  - The SNR at the output of the matched filter is

$$\max\left(\frac{(a_1-a_2)^2}{\sigma_0^2}\right) = \frac{2E_d}{N_0} \qquad E_d = \int_{-\infty}^{+\infty} |S_0(f)|^2\, df$$

  - the maximum output SNR depends on the input signal energy within on symbol and the PSD of noise.
    - It doesn't depend on the particular shape of the waveform



DETECTION: MATCHED FILTER

- **Error probability after matched filter**
  - Error probability

$$P(E) = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$$

  - After matched filter

$$\frac{(a_1 - a_2)^2}{\sigma_0^2} = \frac{2E_d}{N_0} \qquad P(E) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

$$E_d = \int_{-\infty}^{+\infty} |S_0(f)|^2 \, df = \int_0^T [s_1(t) - s_2(t)]^2 \, dt$$

DETECTION: BINARY SYSTEM PERFORMANCE

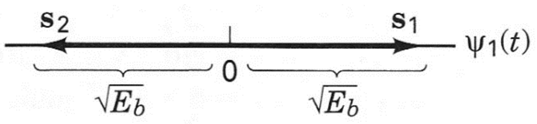
- **Unipolar signaling**
  - $s_1(t) = A \qquad 0 \le t \le T$ for binary '1'

    $s_2(t) = 0 \qquad 0 \le t \le T$ for binary '0'

DETECTION: BINARY SYSTEM PERFORMANCE

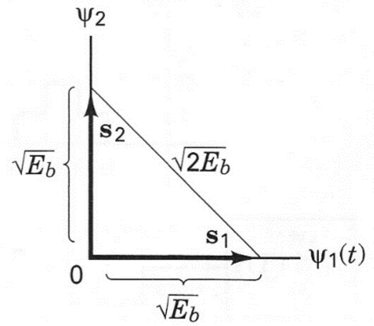
- **Bipolar signaling**
  - $s_1(t) = A \qquad 0 \le t \le T$ for binary '1'

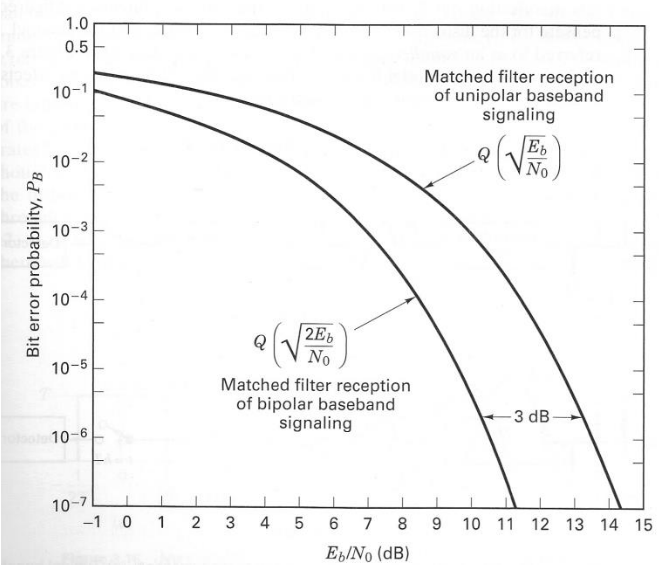
    $s_2(t) = -A \qquad 0 \le t \le T$ for binary '0'



DETECTION: BINARY SYSTEM PERFORMANCE

- **Orthogonal signals**
  - $\int_0^T s_1(t)s_2(t)dt = 0$ $\quad\quad$ $\int_0^T s_1^2(t)dt = \int_0^T s_2^2(t)dt = E_b$





$\sqrt{E_d}$ equals to the Euclidean distance between the constellation points.

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DETECTION: BINARY SYSTEM PERFORMANCE

- **Example**
  - Consider NRZ binary pulses with period T are transmitted along a cable that attenuates signal by 3dB. AWGN with two-sided PSD 10^{-3} watt/Hz. The signal is detected with a matched filter.
    - 1. Assume the signal amplitude is -2 V and 2 V at the transmitter. Determine the maximum data rate that can be sent with a bit error rate (BER) of $P_B \leq 10^{-3}$?
    - 2. If the data rate is 64kbps. What is the signal amplitude required at transmitter to achieve a BER of $P_B \leq 10^{-3}$?

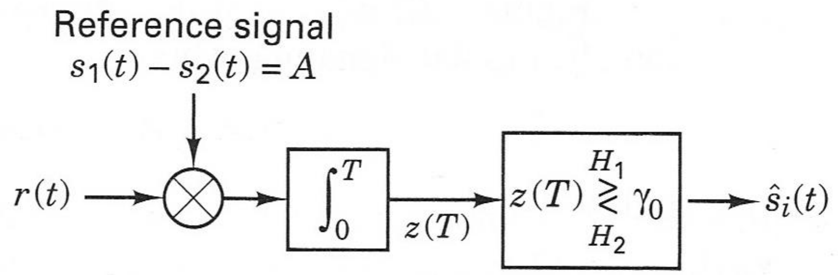
DETECTION: MATCHED FILTER

- **Correlation realization of matched filter**
  - The output of matched filter is

$$z(t) = r(t) \otimes s_0(T - t) =$$

  - The correlation between r(t) and z(t)

$$z_i(T) = \int_0^T r(\tau) s_i(\tau) d\tau$$



DETECTION: BINARY SYSTE PERFORMANCE

- **Graphic interpretation of E_d**
  - Vector signal representation

$$s_i(t) = \sum_{j=1}^{N} a_{ij}\,\psi_j(t)$$

  - $E_d = \int_0^T [s_1(t) - s_2(t)]^2\, dt =$

$$E_d = \sum_{j=1}^{N} (a_{1j} - a_{2j})^2$$

  - $\sqrt{E_d}$ is the distance between two points in an N-dimension Euclidean space

$\mathbf{a}_1 = [a_{11}, \cdots, a_{1N}]$ $\mathbf{a}_2 = [a_{21}, \cdots, a_{2N}]$

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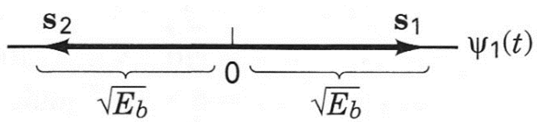
DETECTION: BINARY SYSTE PERFORMANCE

- **Graphic interpretation**
  - 1. unipolar signal

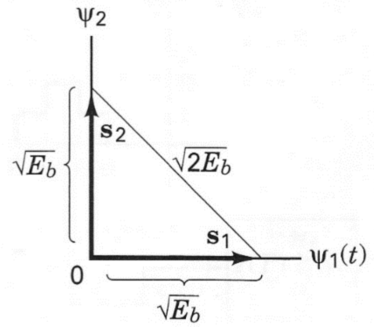
$$s_1(t) = A \qquad 0 \le t \le T$$
$$s_2(t) = 0 \qquad 0 \le t \le T$$

  - 2. bipolar signal

$$s_1(t) = A \qquad 0 \le t \le T$$
$$s_2(t) = -A \qquad 0 \le t \le T$$



  - 3. orthogonal signal



OUTLINE

- Signals and noise
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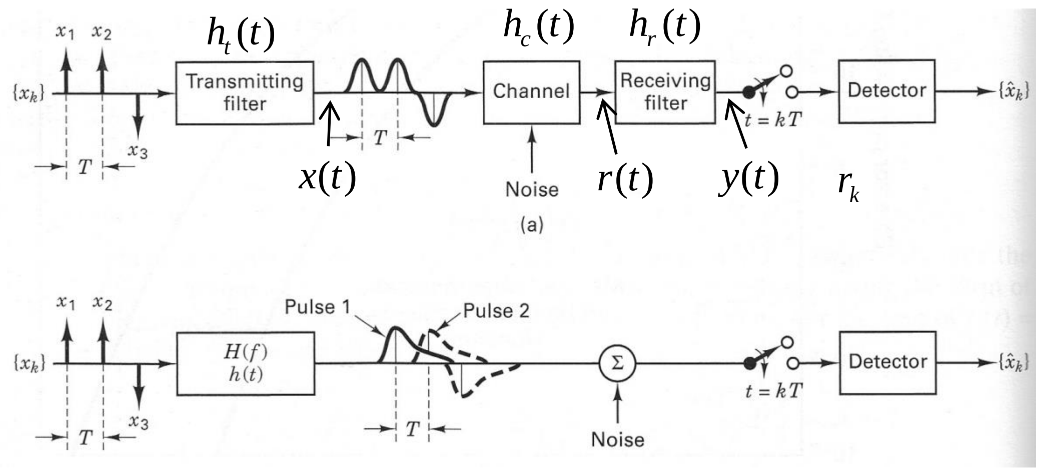
- **Intersymbol Interference (ISI)**
- Equalization

ISI

- **Filters in a communication system**
  - Composite channel impulse response (CIR), or, equivalent filter

$h(t) = h_t(t) \otimes h_c(t) \otimes h_r(t)$ $H(f) = H_t(f) H_c(f) H_r(f)$



CIR $H(f)$

$x(t) =$

$r(t) =$

$y(t) =$

ISI

- **Equivalent Discrete-Time System Model**

$$y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k} + v_n$$

- **Intersymbol interference (ISI)**
  - Due to the effects system filtering, the received pulse can "smear" into adjacent symbol intervals, thereby interfering with other symbols.
  - ISI is introduced by filtering and channel.
  - ISI will degrade system performance even in the absence of noise.

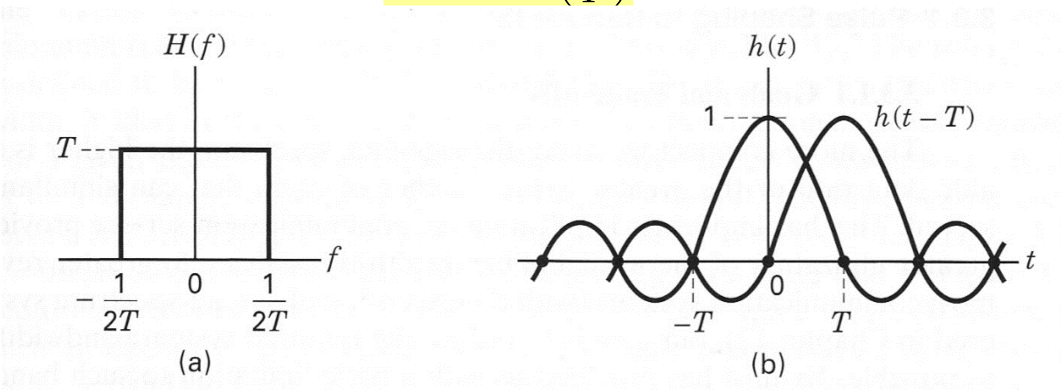
- **Nyquist Criterion**
  - In order to achieve ISI-free communication, the composite impulse response (CIR) must satisfy

$$h_n = h(nT) = \begin{cases} K, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

ISI: NYQUIST FILTER

- **Ideal Nyquist filter**
  - If the CIR is an ideal Nyquist filter, then the symbol can be detected without ISI.

$$h(t) = \sin c\left(\frac{t}{T}\right)$$



- **Nyquist bandwidth constraint**
  - A system with bandwidth *W* Hz can support a maximum symbol rate of

    $R_s = 2W$ symbols/sec without ISI.

    - What is the maximum bandwidth efficiency for M-ary modulation without ISI?

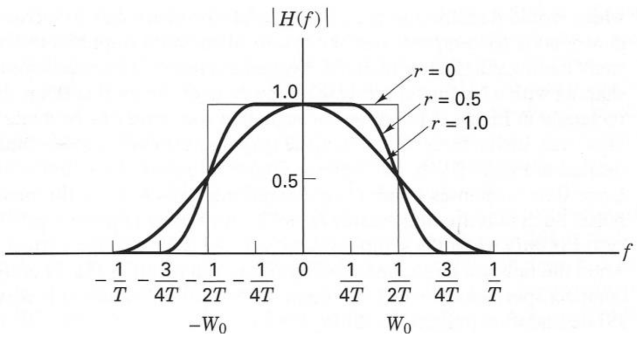
ISI: NYQUIST FILTER

- **Nyquist filter**
  - A general class of filters that satisfy zero ISI at the sampling points
    - Ideal Nyquist filter (rectangular filter, sinc pulse) is one type of Nyquist filter
  - Frequency domain:
    - Ideal nyquist filter $T_s rect(fT_s)$ convolves with any real even-symmetric function leads to a Nyquist filter.
  - Time domain
    - Ideal Nyquist pulse $\sin c(t/T_s)$ multiplied by another time function ➔ Nyquist pulse
  - Most popular Nyquist filter
    - Raised cosine filter

ISI: PULSE SHAPING

- **Pulse shaping**
  - Design transmit filters and receive filters such that the overall CIR, $h(t) = h_t(t) \otimes h_c(t) \otimes h_r(t)$ is a Nyquist pulse ➔ Reduce ISI
  - Limit the bandwidth of the transmitted signal
- **Raised-cosine filter: frequency domain**
  - Frequency domain response

$$H(f) = \begin{cases} 1 & |f| < 2W_0 - W \\ \cos^2\left(\frac{\pi}{4}\frac{|f| + W - 2W_0}{W - W_0}\right) & 2W_0 - W < |f| < W \\ 0 & |f| > W \end{cases}$$



  - W: absolute bandwidth
  - $W_0 = \frac{1}{2T_s}$: minimum Nyquist bandwidth to support a symbol rate of $R_s = \frac{1}{T_s}$
  - $W - W_0$: excessive bandwidth
  - $r = (W - W_0)/W_0$: roll-off factor

$$W = (1+r)W_0 = \frac{1}{2}(1+r)R_s$$

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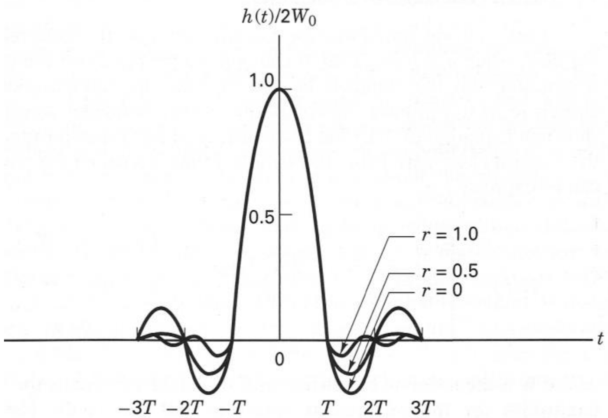
ISI: PULSE SHAPING

- **Raised-cosine filter: time domain**
  - Time domain response

$$h(t) = 2W_0 \sin c(2W_0 t) \frac{\cos[2\pi(W - W_0)t]}{1 - [4(W - W_0)t]^2}$$



- **Raised-cosine filter: roll-off factor**
  - $r = 0$: ideal Nyquist filter
    - Minimum bandwidth in frequency domain: $W = R_s / 2$ ☺
    - Large tail in time domain ➔ a small sampling timing error will introduce large ISI ☹
  - $r = 1$:
    - large bandwidth: $W = R_s$ ☹
    - Small tail in time domain ➔ Less susceptible to sampling timing error ☺

$$W = (1+r)W_0 = \frac{1}{2}(1+r)R_s$$

Baseband bandwidth

$$W_{DSB} = 2(1+r)W_0 = (1+r)R_s$$

Bandpass bandwidth

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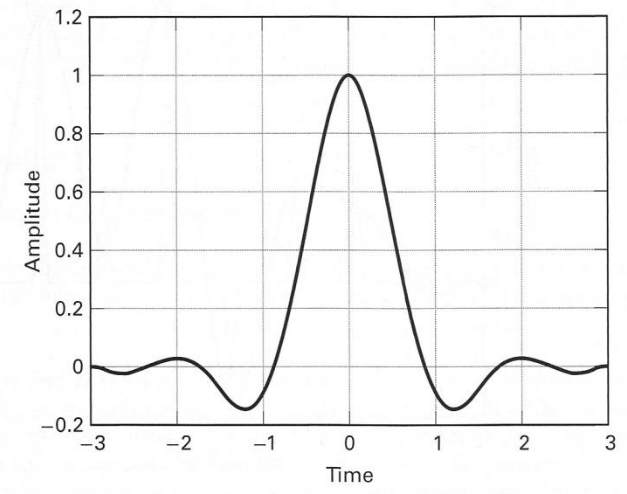
ISI: PULSE SHAPING

- **Square-root raised cosine**
  - Raised cosine is the overall response
    - $h(t) = h_t(t) \otimes h_c(t) \otimes h_r(t)$ is a raised cosine filter
    - $H(f) = H_t(f) H_c(f) H_r(f)$ is a raised cosine filter
  - For AWGN channel $H_c(f) = 1$
  - Square-root raised cosine filter

$$H_t(f) = \sqrt{H(f)}$$

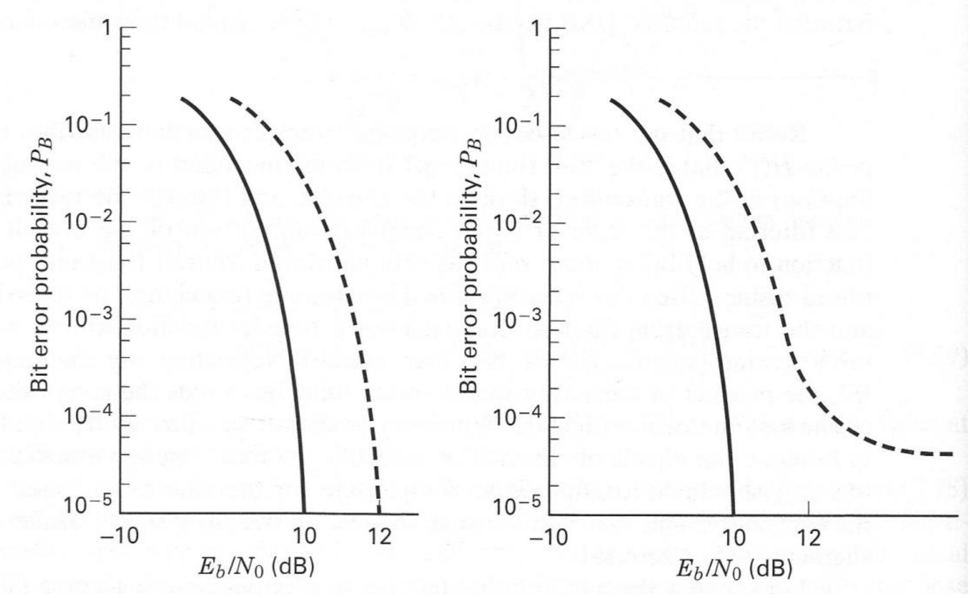
$$H_r(f) = \sqrt{H(f)}$$



$$h_t(t) = \frac{\sqrt{2R_s}}{1 - 64r^2R_s^2t^2}\left[\frac{\sin\left(2\pi R_s(1-r)t\right)}{2\pi R_s t} + \frac{4r}{\pi}\cos\left(2\pi R_s(1+r)t\right)\right]$$

ISI: PERFORMANCE DEGRADATION

- **Two types of performance degradation**
  - 1. Loss in E_b / N_0
    - Can be compensated by increasing signal power
  - 2. Signal distortion
    - E.g. ISI
    - Error floor occurs
    - Cannot be compensated by increasing SNR.



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ISI

- **Example**
  - A 4-level PAM pulse sequence has a data rate of R = 2400 bps. What is the theoretical minimum bandwidth needed for the signal without ISI?
  - If the signal is passed through a raised-cosine filter with 60% excessive bandwidth. Find the bandwidth of the signal at the output of the filter
  - The above sequence is modulated onto a carrier wave so that the baseband spectrum is shifted and centered at frequency f_0 . Find the DSB bandpass bandwidth.

OUTLINE

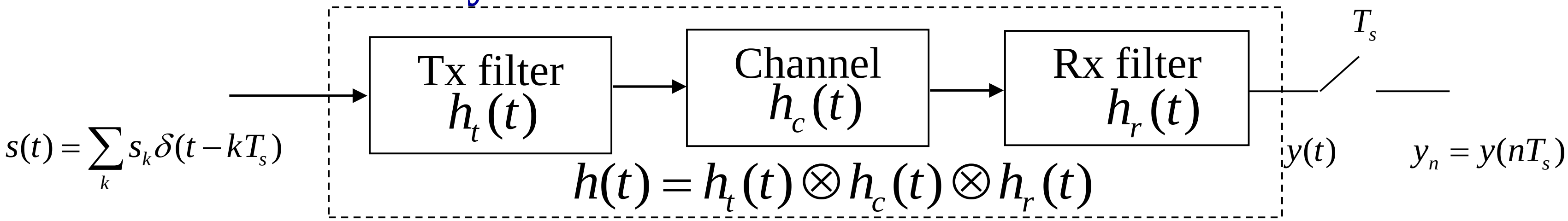
- Signals and noise
- Detection of binary signals in AWGN
- Intersymbol Interference (ISI)
- Matched Filter
- **Equalization**

EQUALIZATION

- **Equalization**
  - Any signal processing or filtering technique that is designed to eliminate or reduce ISI.
  - For AWGN channel, $h_c(t) = \delta(t)$
    - ISI-free communication can be achieved by using square-root raised cosine filter as both transmit filter and receiving filter
      - $h(t) = h_t(t) \otimes \delta(t) \otimes h_r(t)$ is raised cosine (Nyquist filter).
  - For general channel, $h_c(t)$ will introduce ISI
    - Equalization is required.

EQUALIZATION: DISCRETE-TIME MODEL

- **Discrete-time system model**



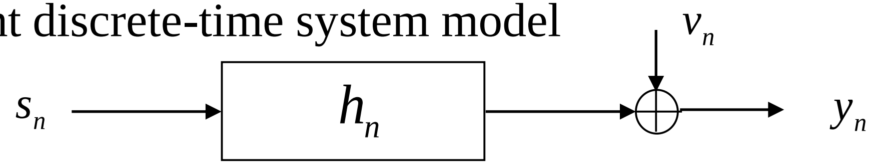
  – Signal after Rx filter

$$y(t)=$$

  – Signal after Sampler

$$y_n=$$

  – Discrete-time CIR $h_n = h(nT_s)$
    - Depends on: 1. Tx filter, 2. Channel, 3. Rx filter, 4. Sampling period
  – Equivalent discrete-time system model



$$y_n = \sum_{l=0}^{L-1} s_{n-l}h_l + v_l = s_n \otimes h_n + v_n$$

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EQUALIZATION: DISCRETE-TIME MODEL

- **Discrete-time system model examples**
  - Example 1: AWGN channel with square root raised cosine filters

$$h_c(t) = \delta(t)$$

$$h(t) = h_t(t) \otimes \delta(t) \otimes h_r(t) = h_{RC}(t)$$

$$h_n = h(nT_s) = \delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

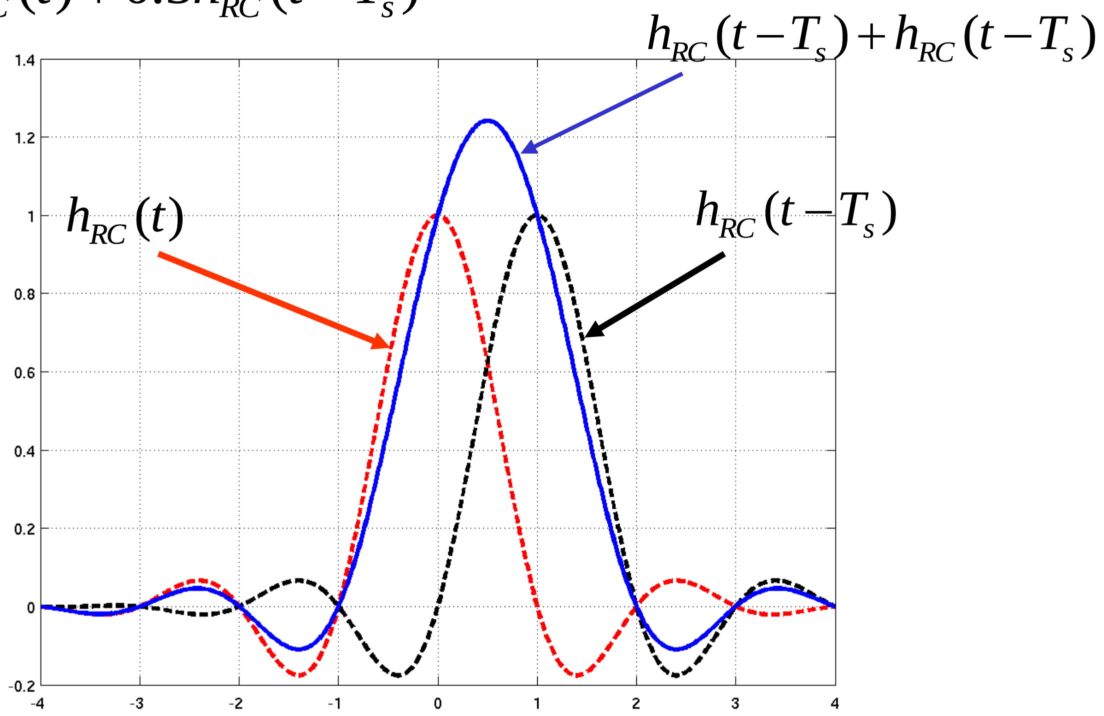
$$y_n = s_n \otimes \delta(n) + v_n = s_n + v_n$$

EQUALIZATION: DISCRETE-TIME MODEL

- **Discrete-time system model examples**
  - Example 2: Two tap equal gain channel

$$h_c(t) = 0.5\delta(t) + 0.5\delta(t - T_s)$$

$$h(t) = h_t(t) \otimes h_c(t) \otimes h_r(t) = h_{RC}(t) \otimes h_c(t)$$
$$= 0.5h_{RC}(t) + 0.5h_{RC}(t - T_s)$$



EQUALIZATION

- **A possible equalization method:**
  - At time 1, $y_1 = h_0 x_1 + 0 + v_1$ (no ISI) ➔ $\hat{x}_1 = y_1 / h_0$
  - At time 2, $y_2 = h_0 x_2 + h_1 x_1 + v_2$ ➔ $\hat{x}_2 = \left[y_2 - h_1 \hat{x}_1 \right] / h_0$
  - At time 3, $y_3 = h_0 x_3 + h_1 x_2 + v_3$ ➔ $\hat{x}_3 = \left[y_3 - h_1 \hat{x}_2 \right] / h_0$
  - Problem: if $\hat{x}_1$ is in error, all the remaining symbols will be affected!
    - Error propagation.

EQUALIZATION: CLASSIFICATION

- **Classification: based on linearity**
  - Linear:
    - Zero-forcing (ZF), minimum mean square error (MMSE)
  - Non-linear
    - Decision feedback equalization (DFE)
    - Maximum likelihood sequence estimation (MLSE) (Optimum)
- **Classification: based on nature of operation**
  - Transversal equalizer
    - ZF, MMSE
    - E.g. Choose equalization filter with frequency response $H_e(f) = \frac{1}{H_c(f)}$

$$H(f) = H_t(f)H_c(f)H_r(f)H_e(f) = H_{RC}(f)$$

  - Transversal filter + feedback filter:
    - DFE
  - Trellis based equalizer
    - MLSE

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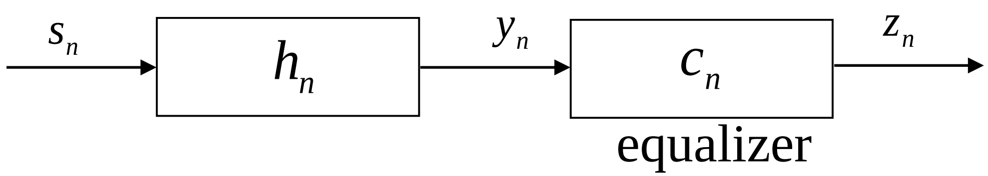
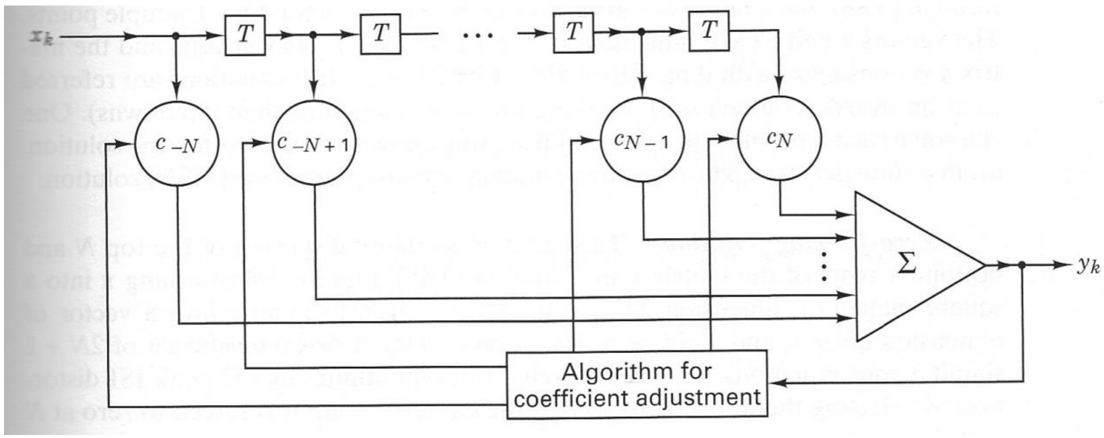
EQUALIZATION: TRANSVERSAL EQUALIZER

- **Transversal equalizer**
  - Suppressing the effects of ISI by passing the received samples through a linear filter





$$z_n = \sum_{k=-N}^{N} c_k y_{n-k}$$

$$n = -N, \cdots, N$$

EQUALIZATION: TRANSVERSAL EQUALIZER

- **Transversal equalizer: Matrix representation**

$$z_n = \sum_{k=-N}^{N} c_k y_{n-k}, \qquad n = -N, \cdots, N$$

$$\begin{bmatrix} z_{-N} \\ \vdots \\ z_0 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} y_0 & y_{-1} & y_{-2} & \cdots & y_{-2N} \\ y_1 & y_0 & y_{-1} & \cdots & y_{-2N+1} \\ & \vdots & & \vdots & \\ y_{2N-1} & \cdots & y_1 & y_0 & y_{-1} \\ y_{2N} & \cdots & y_2 & y_1 & y_0 \end{bmatrix} \begin{bmatrix} c_{-N} \\ \vdots \\ c_0 \\ \vdots \\ c_N \end{bmatrix}$$

$$\mathbf{z} = \mathbf{Yc}$$

  - Example
    - Consider a transversal equalizer with 3 taps (N = 1). The ISI and noise distorted received samples are $y(-3:3) = [0, 0.2, 0.9, -0.3, 0.1]$, represent the transversal equalizer in matrix format

EQUALIZATION: TRANSVERSAL EQUALIZER

- **How do we determine the equalizer coefficients?** $c_n, n = -N, \cdots, N$
  - The operation of equalizer contains two steps: 1. training, 2. transmission
  - Training:
    - Before transmission of the actual data, the transmitter sends a known sequence.
      - Could be a single narrow pulse
      - Or, a pseudo-noise sequence.
    - When the training sequence arrives at receiver, it is distorted by ISI and noise ➔ The distorted training sequence contains information about the channel
    - The receiver calculates the equalizer coefficients based on the ISI and noise distorted training sequence.
      - ZF criterion
      - MMSE criterion
  - Data transmission
    - Once the equalizer coefficients have been trained, the Tx can send actual data.

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EQUALIZATION: ZF

- **Zero-forcing equalization**
  - During the training stage, calculate the transversal equalizer coefficients based on the zero-forcing criterion.
  - Training sequence: $s_n = \delta(n)$
  - Zero-forcing:
    - Selecting $\{c_n\}$ such that the output of the equalizer is $z_n = \delta(n)$
      - Force the elements of z_n to be zero when $n \neq 0$

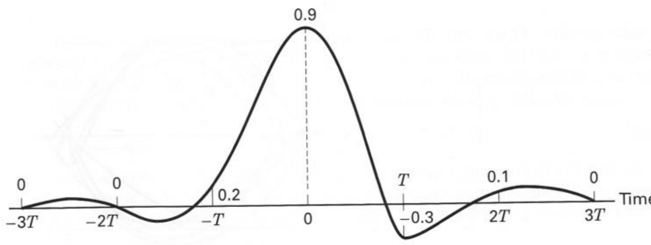
$$\mathbf{z} = [0, \cdots, 0, 1, 0, \cdots, 0]^T$$

$$\mathbf{z} = \mathbf{Y}\mathbf{c}$$

$$\mathbf{c} = \mathbf{Y}^{-1}\mathbf{z}$$

EQUALIZATION: ZF

- **Zero-forcing equalization: example**
  - During the training stage, a impulse is transmitted, and the received distorted samples are $\{y(k)\} = [0.0, 0.2, 0.9, -0.3, 0.1]$
    - Find the zero-forcing equalization coefficients $[c_{-1}, c_0, c_1]$
    - Find the output of the equalizer when the distorted samples are at the input.



EQUALIZATION: MMSE

- **Minimum mean square error (MMSE) equalizer**
  - Minimize the mean square error (variance of error)
  - Training sequence $s_n = \delta(n)$
  - MMSE:
    - Error: $\mathbf{e} = \mathbf{Yc} - \mathbf{z}$
    - MSE: $\varepsilon = E[\mathbf{e}^H \mathbf{e}] = E[(\mathbf{Yc} - \mathbf{z})^H (\mathbf{Yc} - \mathbf{z})]$
    - MMSE: Selecting $\mathbf{c}$ such that the MSE, ε ,is minimized
    - Solution: $\frac{\partial \varepsilon}{\partial \mathbf{c}} = 0$

$$\mathbf{c} = \mathbf{R}_{yy}^{-1} \mathbf{R}_{yz}$$