Department of Electrical Engineering
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ELEG4623/5663 Communication Theory
Ch. 1 Signals and Spectra

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OUTLINE

- **Deterministic signals (Ch. 1.2, 1.3, 1.6 Appendix A)**
- **Random Signals**
- **Signal transmission through linear system, bandwidth**

DETERMINISTIC: SIGNAL CLASSIFICATION

- **Deterministic v.s. Random**
  - Deterministic signal: there is no uncertainty.w.r.t. signal value at any time.
    - E.g. $x(t) = 5\exp(-2t)\cos(5t)$
  - Random signal: there is some degree of uncertainty before the signal occurs.
    - E.g. noise
- **Periodic v.s. Nonperiodic**
  - A signal is called periodic in time if there exists a constant $T_0 > 0$ such that

$$x(t) = x(t + T_0)$$

  - The smallest $T_0 > 0$ satisfying the above equation is called fundamental period.
    - E.g. find the fundamental period of $x(t) = \exp(-j\omega t)$

- **Continuous-time v.s. Discrete-time signal**
  - Continuous-time signal: the signal is defined over continuous-time. $x(t)$
  - Discrete-time signal: a signal that exists only at discrete-time values. $x(kT_s)$, $x(k)$
  - Discrete-time signals are undefined at $t \neq kT_s$

DETERMINISTIC: SIGNAL

- **Energy signal v.s. Power signal**
  - Instantaneous power of a signal $p(t) = x^2(t)$
  - Energy of a signal dissipated during interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$

$$E^T = \int_{-T/2}^{T/2} x^2(t)dt$$

  - Average power dissipated during interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$

$$P^T = \frac{E^T}{T} = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt$$

  - Total Energy of signal: $E = \lim_{T\to\infty}\int_{-T/2}^{T/2} x^2(t)dt = \int_{-\infty}^{\infty} x^2(t)dt$
    - Energy signal: $0 < E < \infty$
  - Average power of signal: $P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt$
    - Average power of periodic sig $P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x^2(t)dt = \frac{1}{T_0}\int_{0}^{T_0} x^2(t)dt$
    - Power signal: $0 < P < \infty$

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DETERMINISTIC: SIGNAL

- **Example:**

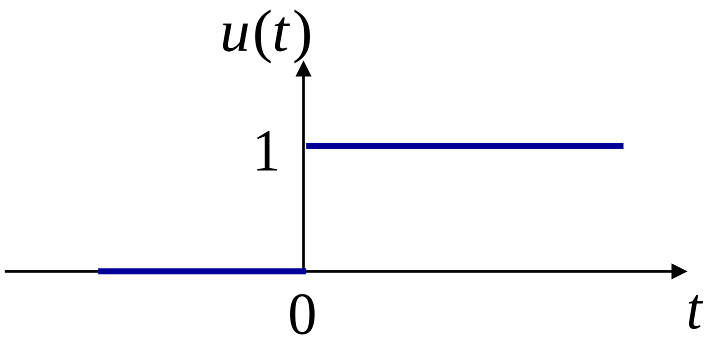
$$x(t) = A\cos(\omega_0 t + \theta_0)$$

  - Find the fundamental period.
  - Find the average power.

DETERMINISTIC: SIGNAL

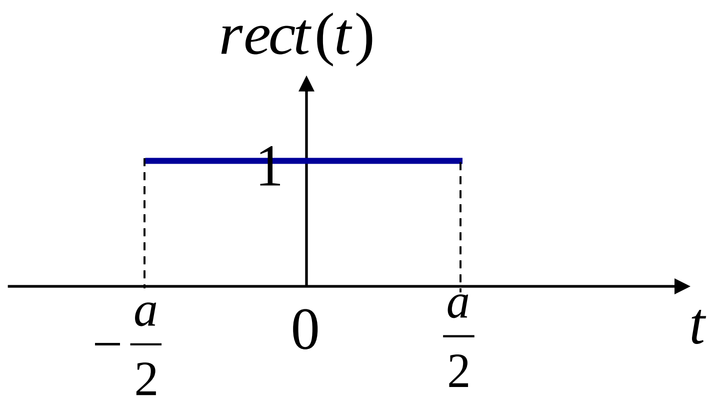
- **Unit step function**

$$u(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$$



- **Rectangular function**

$$rect\left(\frac{t}{a}\right)=\begin{cases}1 & -\frac{a}{2}\leq t\leq\frac{a}{2}\\ 0 & o.w.\end{cases}$$

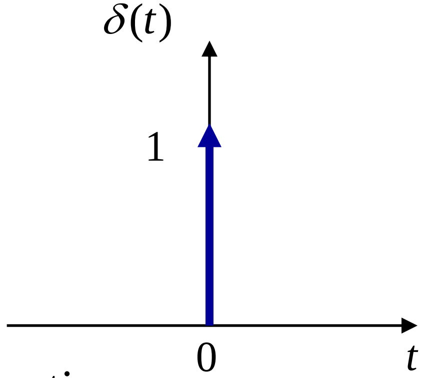


  – *rect*(t) can be represented by using *u*(t):

DETERMINISTIC: SIGNAL

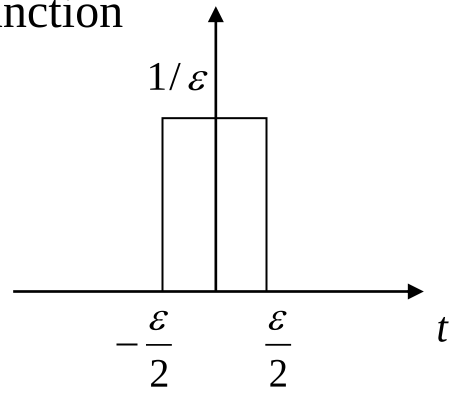
- **Unit impulse function (Dirac delta function)**

$$\delta(0) = \infty$$
$$\delta(t) = 0, t \neq 0$$
$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$



– delta function can be viewed as the limit of the rect function

$$\delta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathrm{rect}(t/\varepsilon)$$



– Sampling property

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

– Shifting property

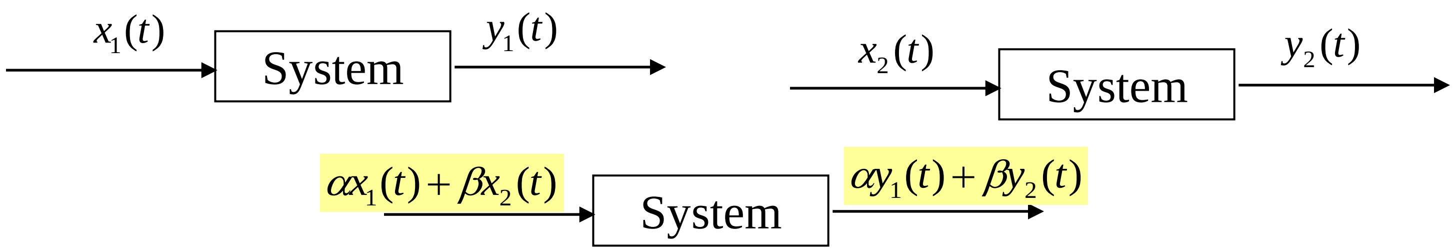
$$\int_{-\infty}^{+\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

– Integration of $\delta(t)$

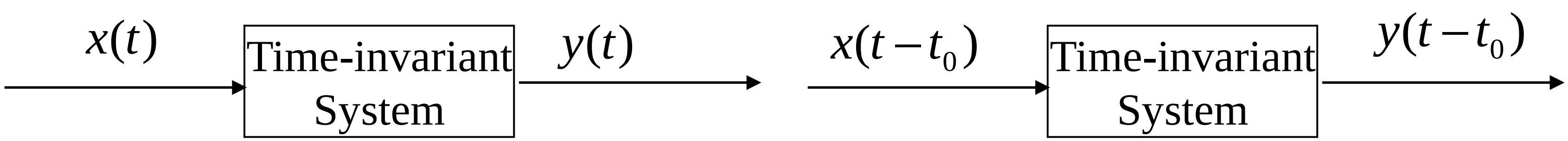
$$\int_{-\infty}^{t} \delta(\tau)d\tau =$$

DETERMINISTIC: SYSTEM

- **Linear system**
  - A system is linear if the superposition principal is satisfied.



- **Time-invariant system**
  - A system is time-invariant if a time shift in the input signal causes an identical time shift in the output signal

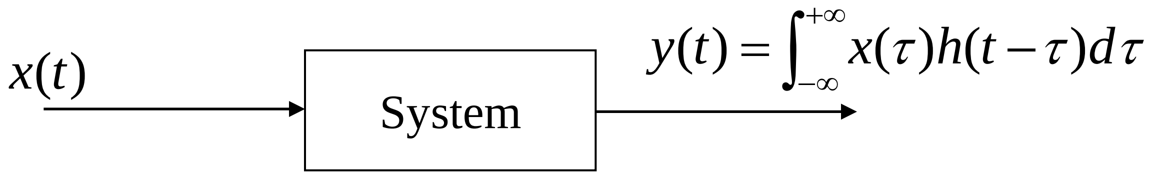


- **Linear time-invariant (LTI) system**
  - A system is both linear and time-invariant.

DETERMINISTIC: SYSTEM

- **Impulse response of LTI system**
  - Def: the output (response) of a system when the input is a unit impulse function (delta function). Usually denoted as $h(t)$

- **Response of LTI system to arbitrary input**



- **Convolution**

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

  - Example: evaluate the convolution $x(t) \otimes \delta(t-t_0)$

DETERMINISTIC: SYSTEM

- **Example**
  - A system has impulse response $h(t) = \exp(-at)u(t)$. If the input is $x(t) = \exp(-bt)u(t)$, find the output.

DETERMINISTIC: FOURIER ANALYSIS

- **Fourier transform**

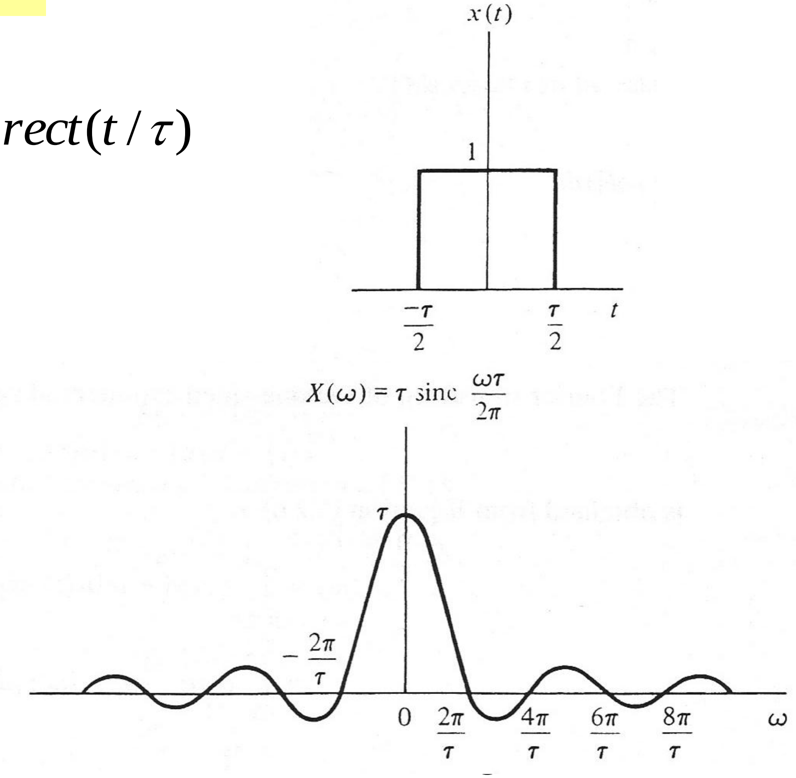
$$X(f)=\int_{-\infty}^{+\infty}x(t)e^{-j2\pi ft}dt$$

  – Frequency domain representation of signal.

- **Inverse Fourier transform**

$$x(t)=\int_{-\infty}^{+\infty}X(f)e^{j2\pi ft}df$$

- **Example:**
  – Find the Fourier transform of $x(t)=rect(t/\tau)$



$$X(f)=\tau\sin c(f\tau)$$

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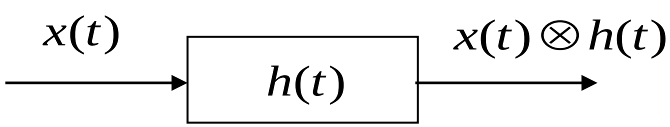
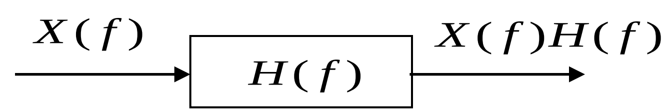
DETERMINISTIC: FOURIER ANALYSIS

- **Selected properties**
  - Linearity
    - If $x_1(t) \Leftrightarrow X_1(f)$ $\quad x_2(t) \Leftrightarrow X_2(f)$
    - Then $ax_1(t) + bx_2(t) \Leftrightarrow aX_1(f) + bX_2(f)$
  - Time shift
    - If $x(t) \Leftrightarrow X(f)$
    - Then $x(t - t_0) \Leftrightarrow X(f)\exp[-j2\pi f t_0]$
  - Duality
    - If $g(t) \Leftrightarrow G(f)$
    - Then $G(t) \Leftrightarrow g(-f)$
  - Convolution
    - If $x(t) \Leftrightarrow X(f)$ $\quad h(t) \Leftrightarrow H(f)$
    - Then $x(t) \otimes h(t) \Leftrightarrow X(f)H(f)$





DETERMINISTIC: FOURIER ANALYSIS

- **Examples**
  - Find the Fourier transform of $x(t) = \delta(t)$
  - Find the Fourier transform of $x(t) = \delta(t - t_0)$
  - Find the Fourier transform of $x(t) = e^{-j2\pi at}$
  - Find the Fourier transform of $x(t) = A\cos(2\pi f_0 t)$

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DETERMINISTIC: FOURIER ANALYSIS

- **Fourier series**
  - For any periodic signal with fundamental period T, it can be decomposed as the sum of a set of complex exponential signals as

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \exp\left[j2\pi \frac{n}{T} t \right]$$

    - Fourier series coefficients: $c_n, n = 0, \pm 1, \pm 2, \cdots$

$$c_n = \frac{1}{T} \int_{<T>} x(t) \exp\left[-j2\pi \frac{n}{T} t \right] dt$$

  - Fourier transform of periodic signal (perform Fourier transform on both sides of Fourier series)

$$X(f) = \sum_{n=-\infty}^{+\infty} c_n \delta\left(f - \frac{n}{T}\right)$$

- **Parsaval's theorem**
  - Energy signal:

$$E = \int_{-\infty}^{+\infty} x^2(t) dt = \int_{-\infty}^{+\infty} X^2(f) df$$

  - Power signal:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt = \sum_{n=-\infty}^{+\infty} c_n^2$$

DETERMINISTIC: ENERGY SPECTRAL DENSITY

- **Energy spectral density (ESD)**
  - The distribution of the signal's energy in frequency domain.
    - The "density" of energy. Unit: Joul/Hz
  - E.g. 1: If the ESD of signal $x(t)$ is $\Psi_x(f)$, then the energy in frequency range $(f, f+\Delta f)$ is:
  - E.g. 2: the energy in frequency range (f_1, f_2) is:
  - Def: If $x(t) \Leftrightarrow X(f)$, then the ESD of energy signal is

$$\Psi_x(f) = |X(f)|^2$$

    - Why?

DETERMINISTIC: POWER SPECTRAL DENSITY

- **Power spectral density (PSD)**
  - The distribution of signals power in frequency domain
    - The density of power (unit: watt/Hz)
  - PSD of power signal is the

$$\Psi_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 \, \delta(f - nf_0)$$

    - c_n: Fourier coefficient
    - $f_0 = 1/T_0$
  - E.g. Find the PSD and power of $x(t) = A\cos(2\pi f_0 t)$

OUTLINE

- Deterministic signals
- Random Signals (Ch. 1.5)
- Signal transmission through linear system, bandwidth

RANDOM SIGNAL

- **Random variable (RV):**
  - Random variable: $X(A)$ represents the functional relationship between a random event A and a number X.
  - Example:
    - Random event A: toss coin;
    - Mapping between coin toss and number:
      - coin head ➔ X = 0; coin tail ➔ X = 1.

- **Discrete RV, probability mass function (PMF)**
  - Example:
    - An urn has 2 black balls, 5 white balls, and 3 red balls, pick one ball out of urn
      - Random event A: black ball, white ball, red ball
    - RV: X: black ball ➔ X = 0; white ball ➔ X = 1; red ball ➔ X = 2.
    - PMF:

$$P(X=0)=$$

$$P(X=1)=$$

$$P(X=2)=$$

RANDOM SIGNAL

- **Cumulative Distribution Function (CDF)**
  - The CDF of a random variable X is given by

$$F_X(x) = P(X \le x)$$

RV   The variable of the function

  - The probability that the RV X is less than or equal to a real number x.
  - Some properties:

$$F_X(-\infty) = 0 \qquad F_X(+\infty) = 1 \qquad 0 \le F_X(x) \le 1$$

$$P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1) \qquad x_1 \le x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$$

  - Example:
    - The CDF of the RV in the previous example

Discrete RV can be characterized by PMF, CDF

RANDOM SIGNAL

- **Probability Density Function (pdf)**

$$p_X(x) = \frac{dF_X(x)}{dx}$$

  - The "density" of probability.
    - E.g. the probability that the RV $X \in [x, x + \Delta x]$:
    - The probability that the RV $X \in [x_1, x_2]$:
  - Properties of pdf

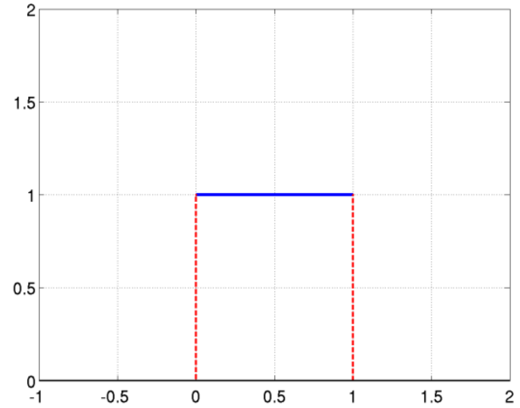
$$p_X(x) \geq 0 \qquad \int_{-\infty}^{+\infty} p_X(x)dx = 1$$

RANDOM SIGNAL

- **Continuous RV**
  - The RV can take continuous values.
  - Continuous RV can be characterized by its pdf or CDF.
- **Uniform distribution**
  - pdf

$$p_X(x) = \frac{1}{b-a}, a \le x \le b$$

  - The RV X has equal probability to be any value in the range of $[a, b]$



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RANDOM SIGNAL

- **Mean (Ensemble Average, 1st moment, expected value)**
  - The mean value of a random variable is defined by

$$m_X = E(X) = \int_{-\infty}^{+\infty} x p_X(x) dx \qquad m_X = E(X) = \sum x_k P(X = x_k)$$

  - Example:
    - The exponential distribution has pdf

$$p_X(x) = \lambda \exp(-\lambda x), x \geq 0, \lambda > 0$$

  - Find its mean value.

RANDOM SIGNAL

- **The n-th moment of a RV is defined as**

$$E(X^n) = \int_{-\infty}^{+\infty} x^n p_X(x)dx$$

- **The n-th central moment of a RV is defined as**

$$E\left[(X - m_X)^n\right] = \int_{-\infty}^{+\infty} (x - m_X)^n p_X(x)dx$$

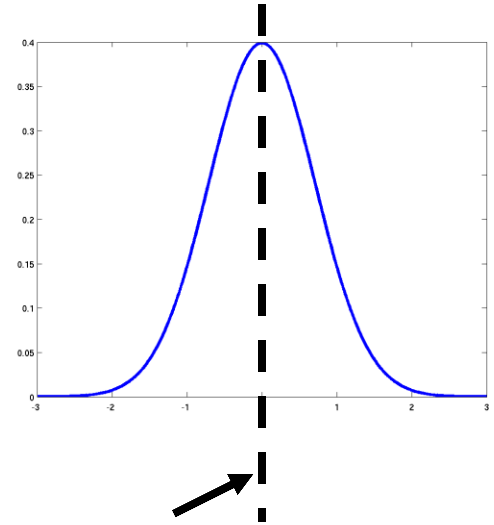
- **Variance (average power of a zero-mean random signal)**
  - Second central moment. $\sigma_X^2 = E\left[(X - m_X)^2\right] = E\left[X^2\right] - m_X^2$
  - Standard deviation (root mean square value, rms): σ_X

RANDOM SIGNAL

- **Gaussian distribution (Normal distribution)**
  - A random variable is Gaussian distributed if the pd

$$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$

  - Gaussian pdf is fully characterized by its mean m and variance σ^2
  - Y = aX+b is still Gaussian
  - Example: prove the mean of Gaussian RV is m



mean $X \sim N(m, \sigma^2)$

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RANDOM SIGNAL: JOINT DISTRIBUTION

- **Joint distribution**
  - Two RVs X, Y, the joint CDF is defined as

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

  - Example

X	Y	Prob.
0	0	0.2
0	1	0.2
1	0	0.5
1	1	0.1

joint PMF: $P(X = 0 \,\&\, Y = 0) =$

marginal PMF: $P(X = 0) =$

marginal PMF: $P(Y = 1) =$

conditional PMF: $P(X = 0 \mid Y = 1) =$

(if we already know that Y = 1, what is the probability that X = 0?)

RANDOM SIGNAL: JOINT DISTRIBUTION

- **Independent RVs**
  - $F_{X,Y}(x,y) = F_X(\mathrm{x})\mathrm{F_Y}(y)$ ←→ X and Y are independent.
  - Independence: there is no relationship between the two RVs.
- **Joint pdf**

$$p_{X,Y}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

- **Marginal pdf**

$$p_X(x) = \int_{-\infty}^{+\infty} p_{XY}(x,y)dy$$

$$p_Y(y) = \int_{-\infty}^{+\infty} p_{XY}(x,y)dx$$

  - $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ ←→ X and Y are independent
- **Conditional pdf**

$$p_{X|Y}(x\,|\,y) = \frac{p_{XY}(x,y)}{p_Y(y)}$$

$$p_{Y|X}(y\,|\,x) = \frac{p_{XY}(x,y)}{p_X(x)}$$

RANDOM SIGNAL: JOINT DISTRIBUTION

- **Example**
  - Find the marginal pdf and conditional pdf of

$$p_{X,Y}(x,y) = e^{-x-y}, x \geq 0, y \geq 0$$

  - Are they independent?

RANDOM SIGNAL: JOINT DISTRIBUTION

- **Correlation and covariance**
  - The correlation between two RVs *X* and *Y*:

$$corr(X,Y) = E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy p_{X,Y}(x,y)dxdy$$

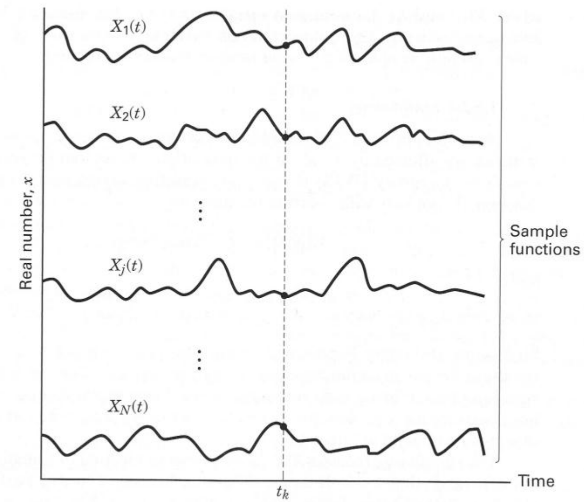
  - The covariance between two RVs *X* and *Y*:

$$\mathrm{cov}(X,Y) = E[(X - m_X)(Y - m_Y)] = E[XY] - m_x m_Y$$

- **Uncorrelated**
  - Two RVs X, Y are uncorrelated if $E[XY] = E[X]E[Y]$
  - If two RVs are uncorrelated, what is their covariance?
  - If two RVs are independent, then they are uncorrelated. (Why?)
    - But not the other way around!!!

RANDOM SIGNAL: RANDOM PROCESS

- **Random process**
  - A random process can be viewed as a RV changes w.r.t. time:
    - A function of two variables: $X(A, t)$
  - Sample function: $X_k(t) = X(A_k, t)$
    - Each sample function corresponds to one of the random events.
    - For a specific event A_k, we have a single sample function $X_k(t)$.
    - The collection of all sample functions is called ensemble.
  - Random variable: $X(t_k) = X(A, t_k)$
    - For a specific time t_k, we have a RV $X(t_k)$
    - Random process is a collection of RVs.



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RANDOM SIGNAL: RANDOM PROCESS

- **Mean (ensemble average)**

$$m_X(t_k) = E\left[X(t_k)\right]$$

  - The mean is a function of time!

- **Autocorrelation function**

$$R_X(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$$

  - The correlation of two RVs.
  - Autocorrelation function is a function of two variables t_1, t_2

- **Stationary (strict)**
  - A random process is stationary in the strict sense if none of its properties is affected by a shift in time.

- **Wide-sense stationary (WSS)**
  - A random process is WSS if its mean and autocorrelation function do not vary with a shift in the time.
    - Mean is independent of time: $m_X(t_k) = m_X$
    - Autocorrelation depends only on time difference: $R_X(t_1, t_2) = R_X(t_1 - t_2)$ $= R_X(\tau)$

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RANDOM SIGNAL: RANDOM PROCESS

- **Example:**

  Consider a stationary sequence of independent binary bits. Each bit has equal probability of being or -1 or 1. The bit period is T.

  – Find the mean of the random process.

  – Find the average power of the random process.

RANDOM SIGNAL: POWER SPECTRAL DENSITY

- **Power spectral density (PSD)**
  - The distribution of the signal's power in the frequency domain.
    - The "density" of power in the frequency domain (unit: watt/Hz).
  - It allows the evaluation of signal power in a certain frequency range.
    - The power in frequency range $[f, f+\Delta f]$:
    - The power in frequency range $[f_1, f_2]$:
  - PSD of a WSS random process is the Fourier transform of its autocorrelation function

$$G_X(f) = F[R_X(\tau)]$$

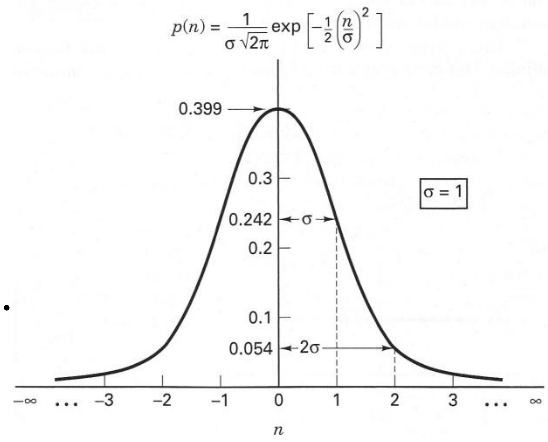
RANDOM SIGNAL: NOISE

- **Noise**
  - Unwanted electrical signals that are always present in electrical system.
    - Man-made noise: spark-plug ignition noise, switching transients, other radiating electromagnetic signal.
    - Natural noise: thermal noise, elements of atmosphere, etc.
- **Thermal noise:**
  - Caused by thermal motion of electrons in all electronic components: resistors, diodes, transistors, wires, …
  - Become worse with the increase of temperature.
  - Thermal noise is a random process X(A, t)

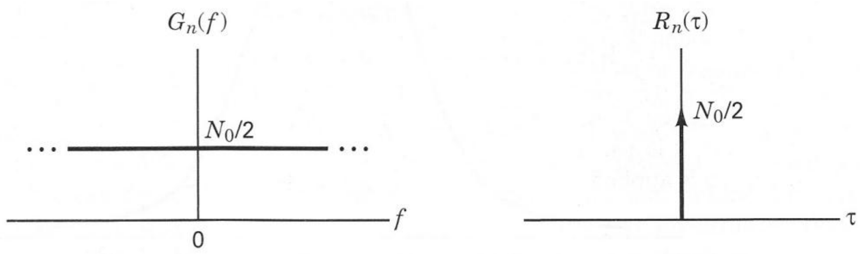
RANDOM SIGNAL: NOISE

- **Statistical properties of thermal noise**
  - At a specific time t_k, $X(A,t_k)$ is zero-mean Gaussian distributed
    - Gaussian noise

$$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right]$$



  - Its power spectral density is the same for all frequencies.
    - White noise
    - PSD $G_X(f) = \frac{N_0}{2}$
    - Autocorrelation function: $R_X(\tau) =$
      - Any two different noise samples are uncorrelated.



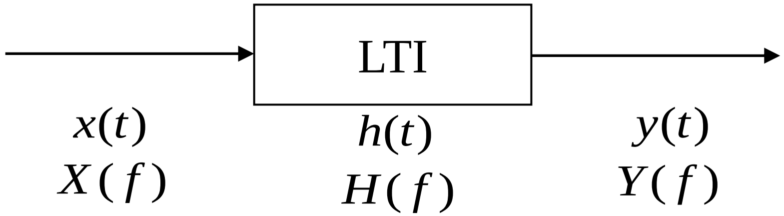
Additive White Gaussian Noise (AWGN)

OUTLINE

- Deterministic signals
- Random Signals
- **Signal transmission through linear system, bandwidth (Ch. 1.6, 1.7)**

TRANSMISSION

- **LTI system**



- **Frequency transfer function (frequency response)**

$$H(f) = \frac{Y(f)}{X(f)} = F\left[h(t)\right]$$

  – In general, $H(f)$ is complex

$$H(f) = |H(f)|\, e^{j\theta(f)}$$

  - Magnitude response:

$$|H(f)| = \sqrt{\mathrm{Re}^2\{H(f)\} + \mathrm{Im}^2\{H(f)\}}$$

  - Phase response:

$$\theta(f) = \arctan \frac{\mathrm{Im}\{H(f)\}}{\mathrm{Re}\{H(f)\}}$$

TRANSMISSION

- **Random process passes through linear system**

$$\xrightarrow[\substack{X(A,t)\\ G_X(f)}]{} \underset{\substack{h(t)\\ H(f)}}{\boxed{\text{LTI}}} \xrightarrow[\substack{Y(A,t)\\ G_Y(f)}]{}$$

  - If the input is a random process $X(A, t)$, then the output is a random process $Y(A, t)$.
  - Generally speaking, Y and X follow different distributions
    - However, if X is Gaussian distributed, Y is still Gaussian!
      - Linear combination of Gaussian is still Gaussian.
  - The PSD of X and Y are related by the following equation

$$G_Y(f) = G_X(f)\left|H(f)\right|^2$$

    - If the input is white Gaussian random process, then the output is colored Gaussian with PSD determined by $H(f)$ ➔ this can be used to generate colored Gaussian random process.

TRANSMISSION: IDEAL TRANSMISSION

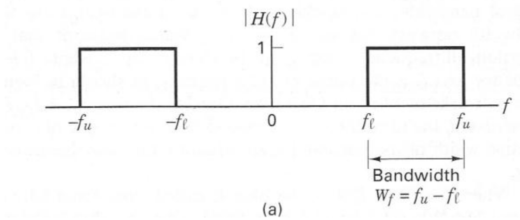
- **Ideal transmission (distortionless transmission)**
  - The output has some delay compared to the input
  - The output has a different amplitude compared to input
  - It must have the same shape as the input: no distortion.

$$y(t) = Kx(t - t_0)$$

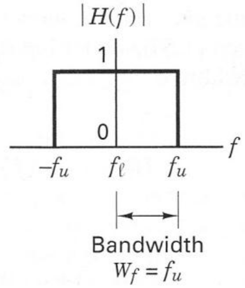
  - Frequency domain system equation:
  - Transfer function:
    - Amplitude response:
    - Phase response:

TRANSMISSION: IDEAL TRANSMISSION

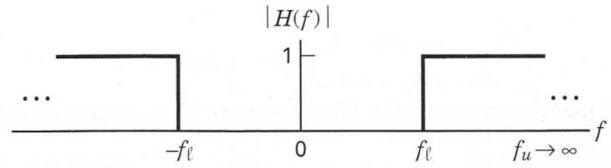
- **Ideal filters**



bandpass



lowpass



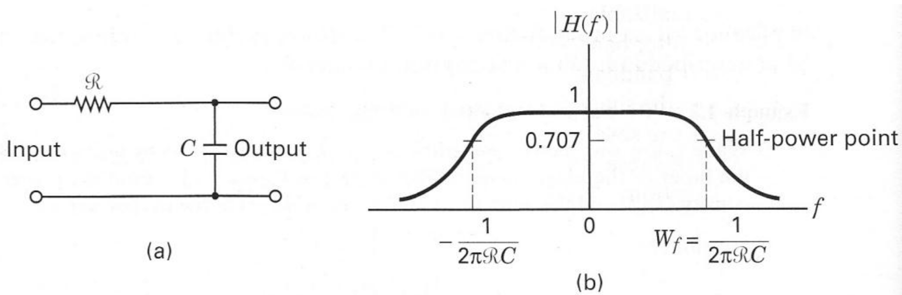
highpass

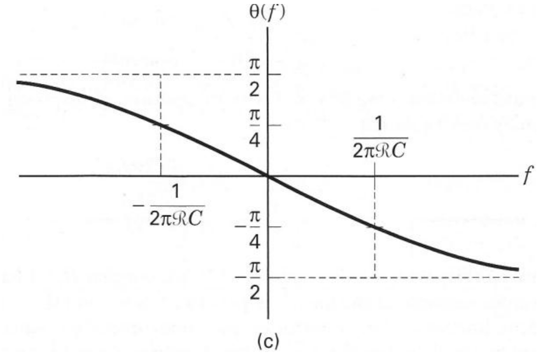
- **Example**
  - Pass a white noise with PSD $G_X(f) = \dfrac{N_0}{2}$ through an ideal low pass filter with bandwidth f_u. Find the autocorrelation function at the output of the filter.

TRANSMISSION: RELIAZABLE TRANSMISSION

- **Example:**
  - RC filter

$$H(f) = \frac{1}{1 + j2\pi fRC}$$





  - Butterworth filter

$$H(f) = \frac{1}{\sqrt{1 + (f/f_u)^{2n}}}, n \geq 1$$

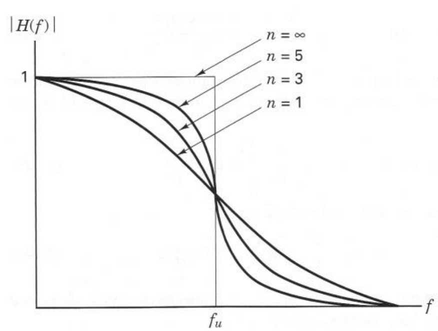


Figure 1.14 Butterworth filter magnitude response.

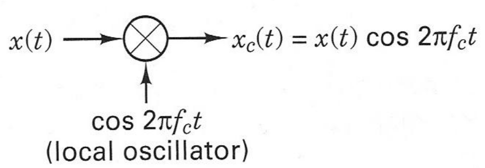
TRANSMISSION: BANDWIDTH

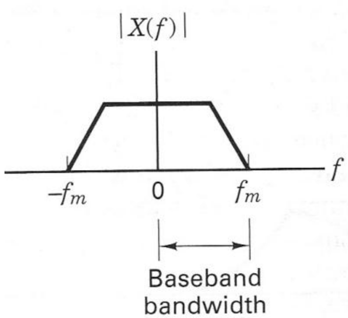
- **Baseband v.s. Bandpass**
  - A baseband signal can be shifted to a higher frequency by multiplying it with a carrier wave $\cos 2\pi f_c t$

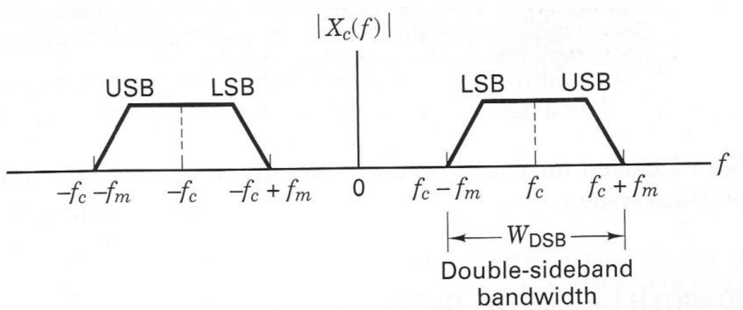
$$x_c(t) = x(t)\cos 2\pi f_c t$$

  - In the frequency domain

$$X_c(f) = \frac{1}{2}\left[X(f - f_c) + X(f + f_c)\right]$$





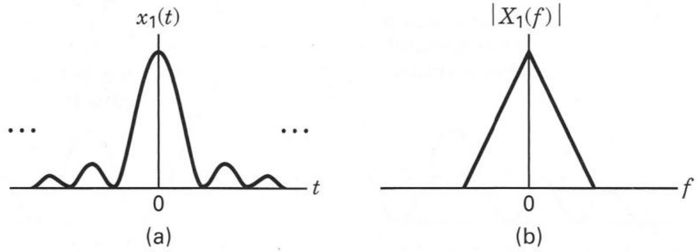


DSB: Double side band, USB: upper side band, LSB: lower side band

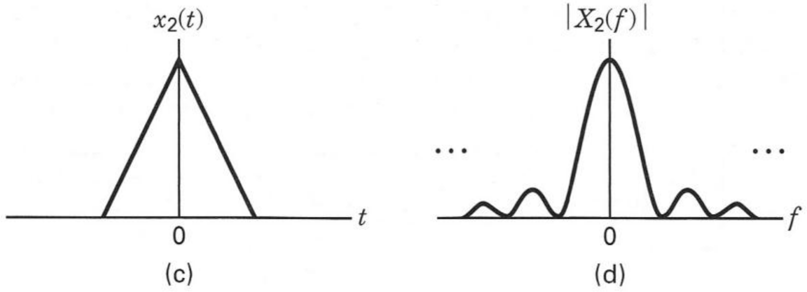
Bandpass bandwidth is twice of baseband bandwidth.

TRANSMISSION: BANDWIDTH DILEMMA

- **Dilemma**
  - Strictly band limited signal imply infinite duration



  - Duration limited signal has infinite bandwidth



A signal cannot be limited in both time domain and frequency domain.

– duration limited signal is realizable →realizable signal is unlimited in frequency domain.

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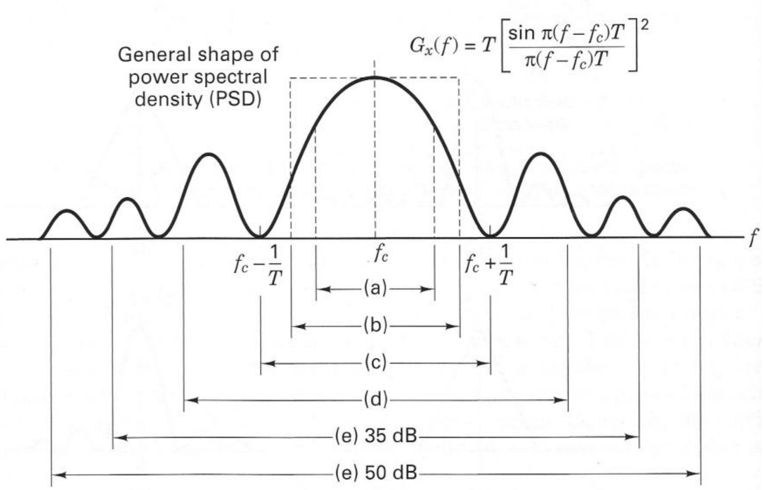
TRANSMISSION: BANDWIDTH DEFINITION



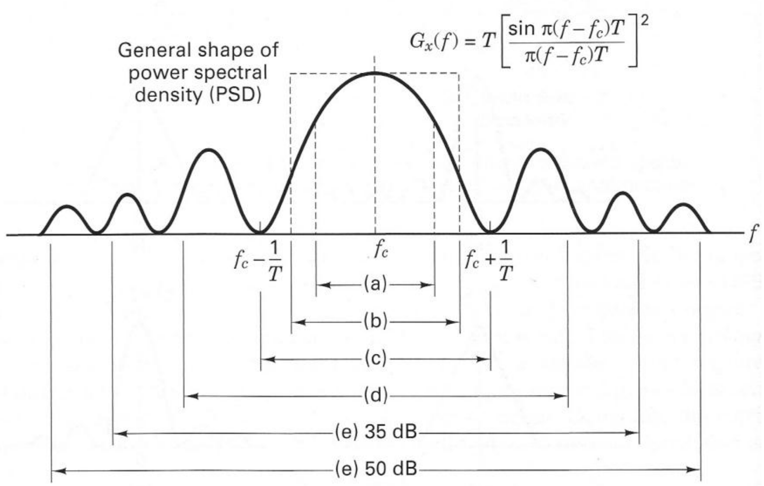
PSD

- (a) Half-power bandwidth:
  - The interval between which that $G(f)$ has dropped to half (3 dB) of its peak value. $G(f_{3dB})/G(f_c)=1/2$
- (b) Equivalent rectangular bandwidth
  - The bandwidth of an equivalent rectangular filter with magnitude the same as the peak of $G(f)$ and has the same total power of $G(f)$

$$W = P/G(f_c)$$

- (c) Null-to-null bandwidth
  - The width of the main spectral lobe.

TRANSMISSION: BANDWIDTH DEFINITION



- (d) Fractional power containment bandwidth (FCC definition)
  - The occupied bandwidth is the band that leaves exactly 0.5% of the signal power above the upper band limit and exactly 0.5% of the signal power below the lower band limit. Thus 99% of the signal power is inside the occupied band.
- (e) bounded power spectral density
  - Everywhere outside the specified band, G(f) must have fallen at least to a specified value (e.g. 35 dB) below its peak value.
- (f) absolute bandwidth:
  - The interval between frequencies, outside of which the PSD is zero.