ELEG 5633 Detection and Estimation
Detection Theory II

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Outline

- Motivation
- Composite Hypothesis Testing
  - Uniformly Most Powerful (UMP) test
  - Bayesian approach
  - Generalized likelihood ratio test (GLRT)

Motivation

So far, we have learned

- Bayesian Detector: given $p(x|H_0)$, $p(x|H_1)$, π_0, π_1, cost assignment
  LRT; MAP; ML
- NP-detector: given $p(x|H_0)$, $p(x|H_1)$
  LRT
- Deterministic signal
  matched filter; generalized matched filter
- Random signal
  energy detector, estimator-correlator.

The distribution of X under H_0, H_1 are known.

What if the PDF of X is not completely known?

Composite Hypothesis Testing

- Assume the PDF of X has unknown parameters under H_0 and H_1

$$\begin{aligned} H_0 :& X \sim p_0(x|\theta_0), \theta_0 \in \Theta_0 \\ H_1 :& X \sim p_1(x|\theta_1), \theta_1 \in \Theta_1. \end{aligned}$$

- Composite hypothesis test: if Θ_0 or Θ_1 contains more than one element.
  - test the hypothesis of more than one models
- Simple hypothesis test: if Θ_m contains a single element, for $m = 0, 1$.

Composite Hypothesis Testing

Three main approaches

- Uniformly Most Powerful (UMP) test
- Bayesian approach
- Generalized likelihood ratio test (GLRT)

Uniformly Most Powerful (UMP) Test

Consider a composite hypothesis test $X \sim p(x|\theta)$, where

$$
\begin{aligned}
H_0 :& \theta = \theta_0 \\
H_1 :& \theta \in \Theta_1.
\end{aligned}
$$

Definition

A UMP test is a hypothesis test that has the greatest power (i.e., P_{D}) $1-\beta$ among all possible tests of a given size (i.e., P_{FA}) α for all $\theta \in \Theta_1$.

- UMP may not exist.
- For a UMP test to exist, the parameter test must be one-sided.

Example

Given N observations, $x[n], n = 0, 1, \ldots, N-1$, which are i.i.d and, depending on the hypothesis, generated according to

$$\begin{aligned} H_0 :& X[n] = W[n] \\ H_1 :& X[n] = A + W[n] \end{aligned}$$

where $W[n] \sim \mathcal{N}(0, \sigma^2)$. The value of A is unknown, although a priori we known $A > 0$ (one-sided) .

Solution: The NP test is to decide H_1 if

$$\frac{p(\mathbf{x}|A, H_1)}{p(\mathbf{x}|H_0)} = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=1}^{N-1}(x[n]-A)^2\right]}{\frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=1}^{N-1}x[n]^2\right]} > \gamma$$

i.e.,

$$A\sum_{n=1}^{N-1} x[n] > \sigma^2 \ln\gamma + \frac{NA^2}{2}$$

Since $A > 0$, we have

$$T(\mathbf{x}) = \frac{1}{N} \sum_{n=1}^{N-1} x[n] > \frac{\sigma^2}{NA} \ln \gamma + \frac{A}{2} = \gamma'$$

Question: Can we implement this detector without knowing the exact value of A?

$$P_{\mathsf{FA}} = \mathbb{P}[T(\mathbf{x}) > \gamma' | H_0] = Q\left(\frac{\gamma'}{\sqrt{\sigma^2/N}}\right)$$

So that

$$\gamma' = \sqrt{\sigma^2/N} Q^{-1}\left(P_{FA}\right)$$

For a given P_{FA}, the threshold of the NP test is independent of A!
Under H_1, $T(\mathbf{x}) \sim \mathcal{N}(A, \sigma^2/N)$, thus

$$P_{\mathsf{D}} = Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/N}}\right) = Q\left(Q^{-1}\left(P_{\mathsf{FA}}\right) - \sqrt{\frac{NA^2}{\sigma^2}}\right)$$

The NP detector yields the highest P_{D} for any value of A, as long as $A > 0$.

UMP Test

Two-sided UMP test does not exist

Consider a two-sided test

$$
\begin{aligned}
H_0 &: A = 0 \\
H_1 &: A \neq 0
\end{aligned}
$$

If $A > 0$, $T(\mathbf{x}) > \sqrt{\sigma^2/N}Q^{-1}\left(P_{\mathsf{FA}}\right)$;
If $A < 0$, $T(\mathbf{x}) < -\sqrt{\sigma^2/N}Q^{-1}\left(P_{\mathsf{FA}}\right)$
Since the value of A is unknown, the NP approach does not result in a unique test.

Suboptimal Tests

When a UMP does not exist, we need to implement suboptimal tests.

- Consider the unknown parameters as realizations of random variables and assign a prior PDF.
  $\Rightarrow$ Bayesian approach
- Estimate the unknown parameters for use in a likelihood ratio test.
  $\Rightarrow$ Generalized likelihood ratio test (GLRT)

Bayesian Approach

- The Bayesian approach assigns prior PDFs to θ_0 and θ_1, denoted by $p(\theta_0)$ and $p(\theta_1)$.

$$p(\mathbf{x}|H_0) = \int p(\mathbf{x}|\theta_0, H_0)p(\theta_0)d\theta_0$$
$$p(\mathbf{x}|H_1) = \int p(\mathbf{x}|\theta_1, H_1)p(\theta_1)d\theta_1$$

- The optimal NP detector decides H_1 if

$$\frac{p(\mathbf{x}; H_1)}{p(\mathbf{x}; H_0)} = \frac{\int p(\mathbf{x}|\theta_1; H_1)p(\theta_1)d\theta_1}{\int p(\mathbf{x}|\theta_0; H_0)p(\theta_0)d\theta_0} > \gamma$$

- It requires multi-dimensional integrations.
- The choice of the prior PDFs can be difficult.

Generalized Likelihood Ratio Test (GLRT)

- The GLRT decides H_1 if

$$L_G(\mathbf{x}) = \frac{\max_{\theta_1 \in \Theta_1} p(\mathbf{x}|\theta_1, H_1)}{\max_{\theta_0 \in \Theta_0} p(\mathbf{x}|\theta_0, H_0)} > \gamma$$

- The first step is to find the maximum likelihood estimation (MLEs) of θ_0 and θ_1.
- The GLRT can be performed in two steps
  - Step 1 (MLE): find the parameter $\theta_m \in \Theta_m$ that can maximize the likelihood function:

$$\hat{\theta}_m(\mathbf{x}) = \underset{\theta_m \in \Theta_m}{\text{argmax}}\, p(\mathbf{x}|\theta_m, H_m)$$

  - Step 2 (LRT): formulate the LRT by using the estimated parameters $\hat{\theta}_0(\mathbf{x})$ and $\hat{\theta}_1(\mathbf{x})$

$$L_G(\mathbf{x}) = \frac{p(\mathbf{x}|\hat{\theta}_1(\mathbf{x}), H_1)}{p(\mathbf{x}|\hat{\theta}_0(\mathbf{x}), H_0)} > \gamma \tag{1}$$

Example

Consider $X[n] = A + W[n]$, $n = 0, 1, \ldots, N-1$, where $W[n]$ is WGN with PDF $\mathcal{N}(0, \sigma^2)$. We have two hypotheses

$$\begin{aligned} H_0 :& A = 0 \\ H_1 :& A \neq 0 \end{aligned}$$

What is the GLRT?

Solution:

- Step 1 (MLE):

$$\hat{A} = \underset{A \neq 0}{\text{argmax}}\, p(\mathbf{x}; A, H_1) = \underset{A \neq 0}{\text{argmin}} \sum_{n=1}^{N} (x[n] - A)^2 = \frac{1}{N} \sum_{n=1}^{N} x[n]$$

- Step 2 (LRT):

$$L_G(\mathbf{x}) = \frac{p(\mathbf{x}|\hat{A}, H_1)}{p(\mathbf{x}|H_0)} > \gamma$$

- Test statistic

$$T(\mathbf{x}) = \sum_{n=1}^{N}(2x[n]\hat{A} - \hat{A}^2) = \frac{1}{N}(\sum_{n=1} x[n])^2 > \gamma'$$

- Under H_0: $Y_N = \sum_{n=1} x[n] \sim \mathcal{N}\left(0, N\sigma^2\right)$, thus

$$H_0 : \left(\frac{Y_N}{\sqrt{N\sigma^2}}\right)^2 = \frac{1}{N\sigma^2}(\sum_{n=1} x[n])^2 = \frac{1}{\sigma^2}T(\mathbf{x}) \sim \chi^2(1)$$

- $$P_{\mathsf{FA}} = \Pr\left(\frac{1}{\sigma^2}T(\mathbf{x}) > \frac{\gamma'}{\sigma^2}|H_0\right) = \Gamma\left(\frac{\gamma'}{2\sigma^2}, \frac{1}{2}\right)$$

$$\gamma' = 2\sigma^2\Gamma^{-1}\left(P_{\mathsf{FA}}, \frac{1}{2}\right)$$