ELEG 5633 Detection and Estimation
Signal Detection: Random Signals

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Outline

- Problem Formulation
- General Gaussian Detection
  - Generalized Matched Filter
  - Energy Detector
  - Estimator-Correlator

Problem Formulation

- Some signals are better represented as random (e.g. speech)
- Rather than assume completely random, assume signal comes from a random process of known covariance structure.

$$
\begin{aligned}
H_0 : & X = W \\
H_1 : & X = S + W
\end{aligned}
$$

  where $W \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w)$, $S \sim \mathcal{N}(\boldsymbol{\mu}_s, \mathbf{C}_s)$
- X has different mean and covariance under different hypothesis.

General Gaussian Detection

- Take LRT,

$$L(\mathbf{x}) = \frac{\frac{1}{(2\pi)^{N/2}|\mathbf{C}_s+\mathbf{C}_w|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_s)^T(\mathbf{C}_s+\mathbf{C}_w)^{-1}(\mathbf{x}-\boldsymbol{\mu}_s)\right]}{\frac{1}{(2\pi)^{N/2}|\mathbf{C}_w|^{1/2}} \exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{x}\right]}$$

- Take the logarithm, retain only the data-dependent terms. The test statistic is

$$T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}_w^{-1}\mathbf{x} - (\mathbf{x}-\boldsymbol{\mu}_s)^T(\mathbf{C}_s+\mathbf{C}_w)^{-1}(\mathbf{x}-\boldsymbol{\mu}_s)$$

General Gaussian Detection (Cont'd)

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$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{x} - (\mathbf{x} - \boldsymbol{\mu}_s)^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} (\mathbf{x} - \boldsymbol{\mu}_s)$$

- Use the matrix inversion lemma

$$\mathbf{C}_w^{-1} - (\mathbf{C}_s + \mathbf{C}_w)^{-1} = \mathbf{C}_w^{-1} \mathbf{C}_s (\mathbf{C}_s + \mathbf{C}_w)^{-1}$$

we have

$$T'(\mathbf{x}) = \underbrace{\mathbf{x}^T (\mathbf{C}_s + \mathbf{C}_w)^{-1} \boldsymbol{\mu}_s}_{\text{linear term}} + \underbrace{\frac{1}{2} \mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{C}_s (\mathbf{C}_s + \mathbf{C}_w)^{-1} \mathbf{x}}_{\text{quadratic term}}$$

- It has a quadratic term in $\mathbf{x}$ (account for the different covariances) and a linear term in $\mathbf{x}$ (account for the different means)

Special Case of General Gaussian Detection

Deterministic Signal: Generalized Matched Filter

- Deterministic Signal: $\mathbf{C}_s = 0$, $S = \boldsymbol{\mu}_s$.

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}_w^{-1} \boldsymbol{\mu}_s$$

Generalized Matched Filter!

Special Case of General Gaussian Detection

Zero-mean White Gaussian Signal in WGN: Energy Detector

- $\boldsymbol{\mu}_s = 0$, $\mathbf{C}_s = \sigma^2 \mathbf{I}_N$, $\mathbf{C}_w = \sigma_w^2 \mathbf{I}_N$

$$H_0 : X \sim \mathcal{N}(\mathbf{0}, \sigma_w^2 \mathbf{I}_N)$$
$$H_1 : X \sim \mathcal{N}(\mathbf{0}, (\sigma^2 + \sigma_w^2)\mathbf{I}_N)$$

  Then,

$$T(\mathbf{x}) = \frac{1}{2} \frac{\sigma^2}{\sigma^2 + \sigma_w^2} \mathbf{x}^T \mathbf{x}$$

  which is equivalent to $T(\mathbf{x}) = \mathbf{x}^T \mathbf{x}$.

- This is known as energy detector, as it computes the energy in the received data.

Special Case of General Gaussian Detection

Zero-mean Correlated Gaussian Signal in WGN: Estimator-Correlator

- $\boldsymbol{\mu}_s = \mathbf{0}$, $S \sim \mathcal{N}(0, \mathbf{C}_s)$

$$
\begin{aligned}
H_0 &: X \sim \mathcal{N}(\mathbf{0}, \sigma_w^2 \mathbf{I}_N) \\
H_1 &: X \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_s + \sigma_w^2 \mathbf{I}_N)
\end{aligned}
$$

  Then,

$$
\begin{aligned}
T(\mathbf{x}) &= \mathbf{x}^T \mathbf{C}_s (\mathbf{C}_s + \sigma_w^2 \mathbf{I}_N)^{-1} \mathbf{x} \\
&= \mathbf{x}^T \hat{S}
\end{aligned}
$$

  where $\hat{S}$ is the MMSE estimator of S.

- This is known as an estimator-correlator.

Canonical Form of the Estimator-Correlator

- When dealing with matched filters with colored noise, we apply a "prewhitening" matrix to white it.
- When the signal has a general convariance matrix $\mathbf{C}_s$, we can de-correlate the received signal first.
- Let the eigen-decomposition of $\mathbf{C}_s$ be $\mathbf{C}_s = \mathbf{V}\Lambda_s\mathbf{V}^T$, where $\Lambda_s = \text{diag}(\lambda_1, \ldots, \lambda_{N-1})$, $\mathbf{V} = [\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{N-1}]$.

$$\begin{aligned} T(\mathbf{x}) &= \mathbf{x}^T\mathbf{C}_s(\mathbf{C}_s + \sigma_w^2\mathbf{I}_N)^{-1}\mathbf{x} \\ &= \mathbf{y}^T\Lambda_s(\Lambda_s + \sigma_w^2\mathbf{I}_N)^{-1}\mathbf{y} \\ &= \sum_{n=0}^{N-1} \frac{\lambda_n}{\lambda_n + \sigma_w^2} y^2[n] \end{aligned}$$

  where $\mathbf{y} = \mathbf{V}^T\mathbf{x}$.
- A weighted energy detector!
- The linear transformation decorrelates $\mathbf{x}$.

Example

Assume $N = 2$, $\mathbf{C}_s = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, where $0 < \rho \leq 1$. $\mathbf{C}_w = \sigma_w^2 \mathbf{I}$. What's the test statistic for a NP detector? What is its canonical form?

Solution:

- The eigenvalues of $\mathbf{C}_s$ are $1 + \rho$, $1 - \rho$.

Special Case of General Gaussian Detection

Correlated Signal in Colored Noise

- $\boldsymbol{\mu}_s = \mathbf{0}$, $S \sim \mathcal{N}(0, \mathbf{C}_s)$, $W \sim \mathcal{N}(0, \mathbf{C}_w)$

$$\begin{aligned} T(\mathbf{x}) &= \mathbf{x}^T \mathbf{C}_w^{-1} \mathbf{C}_s (\mathbf{C}_s + \mathbf{C}_w)^{-1} \mathbf{x} \\ &= \mathbf{x}^T \mathbf{C}_w^{-1} \hat{S} \end{aligned}$$

- $\mathbf{C}_s (\mathbf{C}_s + \mathbf{C}_w)^{-1} \mathbf{x}$ is the MMSE estimator of S.
- Looks like the generalized matched filter for colored noise.