ELEG 5633 Detection and Estimation
Signal Detection: Deterministic Signals

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Outline

- Matched Filter
- Generalized Matched Filter
- Signal Design
- Multiple Signals

Detect a known signal in WGN

- Consider detecting a known deterministic signal in white Gaussian noise. Given a length-N sequence, $x_n, n = 0, 1, \ldots, N-1$, which is generated according to

$$\begin{aligned} H_0 :& X[n] = W[n] \\ H_1 :& X[n] = s[n] + W[n] \end{aligned}$$

  where $W[n] \sim \mathcal{N}(0, \sigma^2)$, $s = \{s[0], \ldots, s[N-1]\}$ is a known and deterministic signal.

NP Detector

- Let $X = [X[0], X[1], \cdots, X[N-1]]^T$.
  Then $H_0 : X \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_n)$, $H_1 : X \sim \mathcal{N}(s, \sigma^2 \mathbf{I}_n)$
- The NP detector is

$$L(x) = \frac{p_1(x)}{p_0(x)} \quad \underset{H_0}{\overset{H_1}{\gtrless}} \quad \gamma$$

$$x^T s \quad \underset{H_0}{\overset{H_1}{\gtrless}} \quad \underbrace{\sigma^2 \ln \gamma + \frac{1}{2}\|s\|^2}_{\gamma'}$$

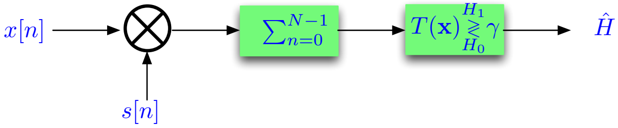
or equivalently

$$T(x) = \sum_{n=0}^{N-1} x[n] s[n] \quad \underset{H_0}{\overset{H_1}{\gtrless}} \quad \gamma'$$

Correlator and Matched Filter

$$T(x) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma'$$

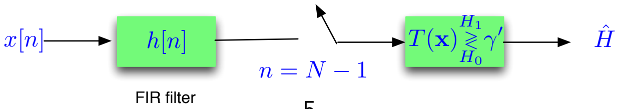
Correlator



Let

$$h[n] = \begin{cases} s[N-1-n] & n = 0, 1, \ldots, N-1 \\ 0 & o.w. \end{cases}$$

Matched Filter



Performance of Matched Filter

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]s[n] > \gamma'$$

Under either hypothesis, $T(X)$ is Gaussian. Let $\mathcal{E} := \sum_{n=0}^{N-1} s^2[n]$,

$$\mathbb{E}[T|H_0] = \mathbb{E}\left[\sum_{n=0}^{N-1} W[n]s[n]\right] = 0$$

$$\mathbb{E}[T|H_1] = \mathbb{E}\left[\sum_{n=0}^{N-1} (s[n] + W[n])s[n]\right] = \mathcal{E}$$

$$\text{var}(T|H_0) = s^T \Sigma_w s = \sigma^2 \sum_{n=0}^{N-1} s^2[n] = \sigma^2 \mathcal{E}$$

$$\text{var}(T|H_1) = s^T \Sigma_w s = \sigma^2 \sum_{n=0}^{N-1} s^2[n] = \sigma^2 \mathcal{E}$$

Performance of Matched Filter (Cont'd)

$$T \sim \begin{cases} \mathcal{N}(0, \sigma^2\mathcal{E}) & \text{under } H_0 \\ \mathcal{N}(\mathcal{E}, \sigma^2\mathcal{E}) & \text{under } H_1 \end{cases}$$

Thus,

$$\begin{aligned} P_{\mathsf{FA}} &= \mathbb{P}[T > \gamma'; H_0] = Q\left(\frac{\gamma'}{\sqrt{\sigma^2\mathcal{E}}}\right) \\ P_{\mathsf{D}} &= \mathbb{P}[T > \gamma'; H_1] = Q\left(\frac{\gamma' - \mathcal{E}}{\sqrt{\sigma^2\mathcal{E}}}\right) \\ &= Q\left(Q^{-1}(P_{\mathsf{FA}}) - \sqrt{\frac{\mathcal{E}}{\sigma^2}}\right) \end{aligned}$$

- Key parameter is the SNR (or energy-to-noise-ratio) $\frac{\mathcal{E}}{\sigma^2}$.
- As SNR increases, P_{D} increases.
- The shape of the signal does NOT affect the detection performance.
- This is not true for colored Gaussian noise.

Generalized Matched Filter

Noise is not i.i.d WGN, but correlated: $W \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$, $\mathbf{C}$ is the covariance matrix. Then,

$$X \sim \begin{cases} \mathcal{N}(\mathbf{0}, \mathbf{C}) & \text{under } H_0 \\ \mathcal{N}(\mathbf{s}, \mathbf{C}) & \text{under } H_1 \end{cases}$$

The optimal test is

$$l(\mathbf{x}) = \ln \frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \gamma$$

where

$$\begin{aligned} l(\mathbf{x}) &= -\frac{1}{2}\left[(\mathbf{x}-\mathbf{s})^T\mathbf{C}^{-1}(\mathbf{x}-\mathbf{s}) - \mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}\right] \\ &= -\frac{1}{2}\left[\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x} - 2\mathbf{x}^T\mathbf{C}^{-1}\mathbf{s} + \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} - \mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}\right] \\ &= \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s} - \frac{1}{2}\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} \end{aligned}$$

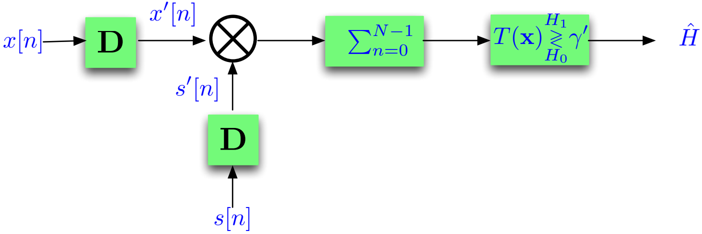
Generalized Matched Filter

Generalized Matched Filter:

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s} \quad \underset{H_0}{\overset{H_1}{\gtrless}} \quad \ln \gamma + \frac{1}{2} \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s} \quad := \gamma'$$

Generalized Matched Filter: Prewhitening Matrix

- When $\mathbf{C} = \mathbf{U}^T\mathbf{A}\mathbf{U}$ is positive definite, $\mathbf{C}^{-1} = \mathbf{U}^T\mathbf{A}^{-1}\mathbf{U}$ exists and is also positive definite, where $\mathbf{A}$ is a diagonal matrix.
- It can be factored as $\mathbf{C}^{-1} = \mathbf{D}^T\mathbf{D}$, where $\mathbf{D} = \mathbf{U}\mathbf{A}^{-1/2}$ is called a prewhitening matrix.
- Then, $T(\mathbf{x}) = \mathbf{x}^T\mathbf{C}^{-1}\mathbf{s} = \mathbf{x}^T\mathbf{D}^T\mathbf{D}\mathbf{s} = \mathbf{x}'^T\mathbf{s}'$, where $\mathbf{x}' = \mathbf{D}\mathbf{x}$, $\mathbf{s}' = \mathbf{D}\mathbf{s}$.



Performance of Generalized Matched Filter

$$T(\mathbf{x}) = \mathbf{x}^T \mathbf{C}^{-1} \mathbf{s} > \gamma'$$

Under either hypothesis, $T(X)$ is Gaussian.

$$\begin{aligned}
\mathbb{E}[T|H_0] &= \mathbb{E}\left[W^T \mathbf{C}^{-1} \mathbf{s}\right] = 0 \\
\mathbb{E}[T|H_1] &= \mathbb{E}\left[(\mathbf{s} + W)^T \mathbf{C}^{-1} \mathbf{s}\right] = \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s} \\
\mathsf{var}(T|H_0) &= \mathbb{E}\left[(W^T \mathbf{C}^{-1} \mathbf{s})^2\right] = \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s} \\
\mathsf{var}(T|H_1) &= \mathsf{var}(T|H_0) = \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}
\end{aligned}$$

$$\begin{aligned}
\text{Thus, } P_{\mathsf{FA}} &= \mathbb{P}[T > \gamma'; H_0] = Q\left(\frac{\gamma'}{\sqrt{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}}}\right) \\
P_{\mathsf{D}} &= \mathbb{P}[T > \gamma'; H_1] = Q\left(\frac{\gamma' - \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}}{\sqrt{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}}}\right) \\
&= Q\left(Q^{-1}(P_{\mathsf{FA}}) - \sqrt{\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}}\right)
\end{aligned}$$

- P_{D} increases with $\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}$, not $\mathcal{E}/\sigma^2$.
- The shape of the signal DOES affect the detection performance.

Signal Design for Correlated Noise

- Objective: design the signal $\mathbf{s}$ to maximize P_{D} for a given P_{FA}, subject to energy constraint $\mathbf{s}^T\mathbf{s} = \mathcal{E}$.

$$\begin{aligned} \text{maximize}_{\mathbf{s}} \quad & \mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} \\ \text{subject to} \quad & \mathbf{s}^T\mathbf{s} = \mathcal{E} \end{aligned}$$

Signal Design for Correlated Noise (Cont'd)

Solution: making use of Lagrangian multipliers

$$F = \mathbf{s}^T \mathbf{C}^{-1} \mathbf{s} + \lambda(\mathcal{E} - \mathbf{s}^T \mathbf{s})$$

Taking derivative w.r.t to $\mathbf{s}$, we have $2\mathbf{C}^{-1}\mathbf{s} - 2\lambda\mathbf{s} = 0$, i.e.,

$$\mathbf{C}^{-1}\mathbf{s} = \lambda\mathbf{s}$$

Thus,

$$\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s} = \lambda \mathbf{s}^T \mathbf{s} = \lambda\mathcal{E}$$

- $\mathbf{s}$ is an eigenvector of $\mathbf{C}^{-1}$ associated with eigenvalue λ.
- To maximize $\mathbf{s}^T \mathbf{C}^{-1} \mathbf{s}$, $\mathbf{s}$ should be associated with the maximum eigenvalue of $\mathbf{C}^{-1}$.
- Since $\mathbf{C}\mathbf{s} = (1/\lambda)\mathbf{s}$, we should choose the signal as the (scaled) eigenvector of $\mathbf{C}$ associated with its minimum eigenvalue.

Example

Assume $W[n] \sim \mathcal{N}(0, \sigma_n^2)$, and W_i's are uncorrelated. How to design the signal $s[n], n = 0, \ldots, N-1$ to maximize P_D?

Example

Assume $\mathbf{C} = \left[\begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right]$, where $0 < \rho \leq 1$. How to design the signal $s[n], n = 0, 1$ to maximize P_D?

Multiple Signals

- Detection

$$\begin{aligned} H_0 : & X = W \\ H_1 : & X = s + W \end{aligned}$$

- Classification

$$\begin{aligned} H_0 : & X = s_0 + W \\ H_1 : & X = s_1 + W \end{aligned}$$

Example

Additive Gaussian White Noise (AWGN) communication channel. Two messages $i = \{0, 1\}$ with probabilities $\pi_0 = \pi_1$. Given i, the received signal is a $N \times 1$ random vector

$$X = s_i + W$$

where $W \sim \mathcal{N}(0, \sigma^2 I_{N \times N})$. $\{s_0, s_1\}$ are known to the receiver. Design the ML receiver.

Minimum Distance Receiver

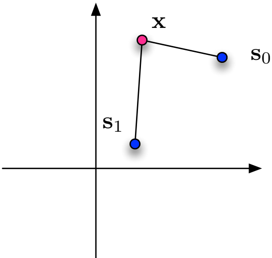
We want to minimize P_e, with equal prior probabilities for H_i $\Rightarrow$ ML detector

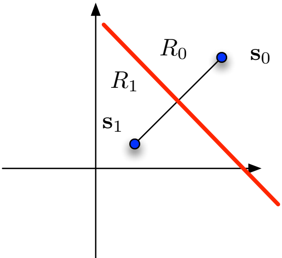
$$p(\mathbf{x}|H_i) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-s_i[n])^2\right]$$

Decide H_i for which

$$D_i^2 := \sum_{n=0}^{N-1}(x[n]-s_i[n])^2 = (\mathbf{x}-\mathbf{s}_i)^T(\mathbf{x}-\mathbf{s}_i) = \|\mathbf{x}-\mathbf{s}_i\|^2$$

is minimum. $\Rightarrow$ minimum distance receiver





Minimum Distance Receiver

$$D_i^2 := \sum_{n=0}^{N-1} (x[n] - s_i[n])^2 = \sum_{n=0}^{N-1} x^2[n] - 2 \sum_{n=0}^{N-1} x[n]s_i[n] + \sum_{n=0}^{N-1} s_i^2[n]$$

Therefore, we decide H_i for which

$$T_i(\mathbf{x}) := \sum_{n=0}^{N-1} x[n]s_i[n] - \frac{1}{2} \sum_{n=0}^{N-1} s_i^2[n] = \sum_{n=0}^{N-1} x[n]s_i[n] - \frac{1}{2}\mathcal{E}_i$$

is maximum.

Binary Case

- ML detector

$$T_1(\mathbf{x}) - T_0(\mathbf{x}) = \sum_{n=0}^{N-1} x[n](s_1[n] - s_0[n]) - \frac{1}{2}(\mathcal{E}_1 - \mathcal{E}_0) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

or equivalently

$$\mathbf{x}^T(\mathbf{s}_1 - \mathbf{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}(\mathcal{E}_1 - \mathcal{E}_0)$$

- Error probability

$$P_e = Q\left(\frac{1}{2}\frac{\|\mathbf{s}_1 - \mathbf{s}_0\|}{\sigma}\right)$$

Performance for Binary Case

$$
\begin{aligned}
P_e &= \mathbb{P}[\hat{H} = H_1 | H_0]\pi_0 + \mathbb{P}[\hat{H} = H_0 | H_1]\pi_1 \\
&= \frac{1}{2}\left(\mathbb{P}[T_1(X) - T_0(X) > 0 | H_0] + \mathbb{P}[T_1(X) - T_0(X) < 0 | H_1]\right)
\end{aligned}
$$

Let $\quad T(\mathbf{x}) := T_1(\mathbf{x}) - T_0(\mathbf{x}) = \sum_{n=0}^{N-1} x[n](s_1[n] - s_0[n]) - \frac{1}{2}(\mathcal{E}_1 - \mathcal{E}_0)$

T is a Gaussian random variable conditioned on either hypothesis!

$$\mathbb{E}[T|H_0] = \sum_{n=0}^{N-1} s_0[n](s_1[n] - s_0[n]) - \frac{1}{2}(\mathcal{E}_1 - \mathcal{E}_0) = -\frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_0\|^2$$

$$\mathbb{E}[T|H_1] = -\mathbb{E}[T|H_0] = \frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_0\|^2$$

$$\mathsf{var}(T|H_0) = \sum_{n=0}^{N-1} \mathsf{var}(x[n])(s_1[n] - s_0[n])^2 = \sigma^2\|\mathbf{s}_1 - \mathbf{s}_0\|^2 = \mathsf{var}(T|H_1)$$

$$T \sim \begin{cases} \mathcal{N}(-\frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_0\|^2, \sigma^2\|\mathbf{s}_1 - \mathbf{s}_0\|^2) & H_0 \\ \mathcal{N}(\frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_0\|^2, \sigma^2\|\mathbf{s}_1 - \mathbf{s}_0\|^2) & H_1 \end{cases} \qquad P_e = Q\left(\frac{1}{2}\frac{\|\mathbf{s}_1 - \mathbf{s}_0\|}{\sigma}\right)$$