ELEG 4603/5173L Digital Signal Processing
Ch. 1 Discrete-Time Signals and Systems

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OUTLINE

• Classifications of discrete-time signals

• Elementary discrete-time signals

• Linear time-invariant (LTI) discrete-time systems

• Causality and stability

• Difference equation representation of LTI systems
• **Discrete-time signal**
  
  -- A signal that is defined only at discrete instants of time.
  -- Represented as $x(n)$, $n = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$

\[
x(n) = \cos\left(\frac{n}{4}\right)
\]

\[
x(n) = \frac{1}{2} \exp\left(\frac{n}{4}\right)
\]

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SIGNAL CLASSIFICATION
• **Review: Analog v.s. digital**
  – Continuous-time signal $x(t)$,
    • continuous-time, continuous amplitude $\rightarrow$ **analog signal**
      – Example: speech signal
    • Continuous-time, discrete amplitude
      – Example: traffic light
  – Discrete-time signal $x(n)$,
    • Discrete-time, discrete-amplitude $\rightarrow$ **digital signal**
      – Example: Telegraph, text, roll a dice
    • Discrete-time, continuous-amplitude
      – Example: samples of analog signal, average monthly temperature
SIGNAL CLASSIFICATION

- Periodic signal v.s. aperiodic signal
  - Periodic signal \( x(n) = x(n + N) \)
    - The smallest value of \( N \) that satisfies this relation is the fundamental periods.
  - Is \( \cos(\omega n) \) periodic?

\[ \cos(\omega n) \text{ is periodic if } \frac{2k\pi}{\omega} = N \text{ is an integer, and the smallest } N \text{ is the fundamental period.} \]

- Example: \( \cos(3n) \)

\[
\begin{align*}
\cos(\pi n) \\
\cos\left(\frac{3}{4} \pi n\right)
\end{align*}
\]
SIGNAL CLASSIFICATION

- Sum of two periodic signals

\[ x_1(n) : \text{fundamental period} \quad N_1 \]
\[ x_2(n) : \text{fundamental period} \quad N_2 \]
\[ x_1(n) + x_2(n) \]

\[ x_1(n) + x_2(n) \] is periodic if both \( x_1(n) \) and \( x_2(n) \) are periodic. Assume

\[ \frac{N_1}{N_2} = \frac{p}{q} \]

where \( p \) and \( q \) are not divisible of each other. The period is \( N = pN_2 = qN_1 \)
**SIGNAL CLASSIFICATION**

- **Example:**
  - Is the signal periodic? If it is, what is the fundamental period?

\[
\cos\left(\frac{\pi n}{9}\right) + \sin\left(\frac{3\pi n}{7} + \frac{1}{2}\right)
\]
SIGNAL CLASSIFICATION

- **Energy signal**
  - Energy:
    \[
    E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |x(n)|^2
    \]
  - Review: energy of continuous-time signal
    \[
    E = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt
    \]
  - Energy signal: \(0 < E < \infty\)
SIGNAL CLASSIFICATION

- **Power signal**
  - Power of discrete-time signal
    \[
    P = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} |x(n)|^2
    \]
  - Review: power of continuous-time signal
    \[
    P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt
    \]
  - Power signal: \(0 < P < \infty\)
SIGNAL CLASSIFICATION

• **Example**
  
  – Determine if the discrete-time exponential signal is an energy signal or power signal

  \[ x(n) = 2(0.5)^n \quad n \geq 0 \]
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**ELEMENTARY SIGNALS**

- **Basic signal operations**
  - Time shifting $x(n - k)$
    - shift the signal to the right by $k$ samples
  
  ![Graphs of $x(n)$, $x(n+3)$, and $x(n-3)$](images)

- Reflecting $x(-n)$
  - Reflecting $x(n)$ with respect to $n = 0$.

  ![Graphs of $x(n)$ and $x(-n)$](images)
ELEMENTARY SIGNALS

- **Basic signal operations**
  - Time scaling
  - Example: Let \( x(n) = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases} \)

Find \( x(2n) \), \( x(2n + 1) \)
ELEMENTARY SIGNALS

• Basic signal operations
  – Time scaling
    • Example: \( x(n) = [-1, 2, 1, 0, -2] \)

\[\Rightarrow \text{the always points to } x(0)\]

\[
\text{find } x(3n), x\left(\frac{n}{3}\right), x\left(\frac{n}{3} + \frac{2}{3}\right)
\]
ELEMENTARY SIGNALS

- **Unit impulse function**
  \[ \delta(n) = \begin{cases} 
  1, & n = 0, \\
  0, & n \neq 0. 
\end{cases} \]

  - time shifting
  \[ \delta(n - k) = \begin{cases} 
  1, & n = k, \\
  0, & n \neq k. 
\end{cases} \]

- **Unit step function**
  \[ u(n) = \begin{cases} 
  0, & n < 0, \\
  1, & n \geq 0. 
\end{cases} \]

- **Relation between unit impulse function and unit step function**
  \[ \delta(n) = u(n) - u(n-1) \]

  \[ u(n) = \sum_{k=-\infty}^{n} \delta(k) \]
Elementary Signals

- **Exponential function**

  \[ x(n) = \exp(\alpha n) \]

- **Complex exponential function**

  \[ x(n) = \exp(j\omega_0 n) = \cos(\omega_0 n) + j \sin(\omega_0 n) \]

  - \( x(n) \) is periodic if \( \frac{2k\pi}{\omega_0} = N \) is an integer, and the smallest integer \( N \) is the fundamental period.

- **Example**

  - Are the following signals periodic? If periodic, find fundamental period.

    \[ x_1(n) = \exp\left(j\frac{7\pi}{9}n\right) \quad x_2(n) = \exp\left(j\frac{7}{9}n\right) \]
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DISCRETE-TIME SYSTEMS

• **Linear system**
  - Consider a system with the following input-output relationship

  \[ x_1(n) \rightarrow \text{System} \rightarrow y_1(n) \]
  \[ x_2(n) \rightarrow \text{System} \rightarrow y_2(n) \]

  - The system is linear if it meets the superposition principle

  \[ \alpha x_1(n) + \beta x_2(n) \rightarrow \text{System} \rightarrow \alpha y_1(n) + \beta y_2(n) \]

• **Time-invariant system**
  - Consider a system with the following input-output relationship

  \[ x(n) \rightarrow \text{System} \rightarrow y(n) \]

  - The system is time-invariant if a time-shift at the input leads to the same time-shift at the output

  \[ x(n-k) \rightarrow \text{System} \rightarrow y(n-k) \]
Any arbitrary discrete-time signal can be decomposed as weighted summation of the unit impulse functions

\[ x(n) = \sum_{k=-\infty}^{+\infty} x(k) \delta(n - k) \]

- Why?
  E.g. \( x(3) = \sum_{k=-\infty}^{+\infty} x(k) \delta(3 - k) = \)

- Recall: \( x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau \)
• **LTI response to arbitrary input**
  - Any arbitrary signal can be written as
  \[
x(n) = \sum_{k=\infty}^{+\infty} x(k) \delta(n-k)
  \]
  - Time-invariant
  \[
  \delta(n-k) \xrightarrow{\text{LTI}} h(n-k)
  \]
  - Linear
  \[
  \sum_{k=\infty}^{+\infty} x(k) \delta(n-k) \xrightarrow{\text{LTI}} \sum_{k=\infty}^{+\infty} x(k)h(n-k)
  \]
  \[
  x(n) \xrightarrow{\text{LTI}} y(n) = \sum_{k=\infty}^{+\infty} x(k)h(n-k)
  \]
DISCRETE-TIME SYSTEM

- **Convolution sum**
  - The convolution sum of two signals $x(n)$ and $h(n)$ is

  $$x(n) \otimes h(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

- **Response of LTI system**
  - The output of a LTI system is the convolution sum of the input and the impulse response of the system.
Examples

1. \( x(n) \otimes \delta(n - m) \)

2. \( x(n) = \alpha^n u(n), \quad h(n) = \beta^n u(n) \)
\[ x(n) \otimes h(n) = \]
Example

3. Find the step response of the system with impulse response

\[ h(n) = 2 \left( \frac{1}{2} \right)^n \cos \left( \frac{2\pi}{3} n \right) u(n) \]
Example:

Let \( x(n) = [0,1,2] \) and \( h(n) = [-1,-2,-3,-4] \), be two sequences, find \( x(n) \otimes h(n) \).
DISCRETE-TIME SYSTEM

• Properties: commutativity

\[ x(n) \otimes h(n) = h(n) \otimes x(n) \]
• Properties: associativity

\[ x(n) \otimes h_1(n) \otimes h_2(n) = [x(n) \otimes h_1(n)] \otimes h_2(n) = x(n) \otimes [h_1(n) \otimes h_2(n)] \]
• Distributivity

\[ x(n) \otimes [h_1(n) + h_2(n)] = [x(n) \otimes h_1(n)] + [x(n) \otimes h_1(n)] \]
• Example
  
  Consider a system shown in the figure. Find the overall impulse response.

  $$h_1(n) = \delta(n) - 2\delta(n-1) \quad h_2(n) = (n-1)u(n) \quad h_3(n) = 2^n u(n)$$
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CAUSALITY AND STABILITY

- **Causal system**
  - A discrete-time system is causal if the output \( y(n_0) \) depends only on values of input for \( n \leq n_0 \)
  - The output does not depend on future input.
  - Example:
    - determine whether the following systems are causal.
      \[
      y(n) = x^2(n) + 3x(n)
      \]
      \[
      y(n) = x^2(n-1)
      \]
      \[
      y(n) = \frac{1}{3} [x(n-1) + x(n) + x(n+1)]
      \]
      \[
      y(n) = \sum_{k=-\infty}^{n} x(k)
      \]
      \[
      y(n) = \sum_{k=-\infty}^{n+1} x(k)
      \]
CAUSALITY AND STABILITY

- **Causality of LTI system**
  - An LTI system is causal if and only if $h(n)=0$ for $n < 0$
  - Why?
    \[
    y(n) = \sum_{-\infty}^{+\infty} x(k)h(n-k) = \sum_{-\infty}^{n} x(k)h(n-k) + \sum_{n+1}^{+\infty} x(k)h(n-k)
    \]
  - A signal $x(n)$ is causal if $x(n)=0$ for $n < 0$.
  - Example:
    - For LTI systems with impulse responses given as follows. Find if the systems are casual.
      \[
      h(n) = \cos(2n)
      \]
      \[
      h(n) = \cos(2n)u(n)
      \]
      \[
      h(n) = a^n u(n) + b^n u(n+1)
      \]
      \[
      h(n) = a^n u(n) + b^n u(n-1)
      \]
CAUSALITY AND STABILITY

• Bounded-input bounded-output (BIBO) stable
  – a system is BIBO stable if, for any bounded input $x(n)$, the response $y(n)$ is also bounded.

• BIBO stability of LTI system
  – An LTI discrete-time system is BIBO stable if

\[ \sum_{-\infty}^{+\infty} |h(k)| < \infty \]

• Why?
Example

For an LTI system with impulse responses as follows. Are they BIBO stable?

\[ h(n) = (0.5)^n u(n) \]

\[ h(n) = (-0.5)^n u(n) \]

\[ h(n) = (0.5)^n \]

\[ h(n) = 2^n u(n) \]

\[ h(n) = 2^n u(-n) \]
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• Difference equation representation of LTI discrete-time system
  – Any LTI discrete-time system can be represented as

\[ \sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \]

– Review: any LTI continuous-time system can be represented as

\[ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \]
DIFFERENCE EQUATION

- Simulation diagram

\[ y(n) + a_1 y(n-1) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \cdots + b_N x(n-N) \]
Example

Draw the simulation diagram of the LTI system described by the following difference equation

$$2y(n) + 3y(n-1) = 0.5x(n) + x(n-1) + 5x(n-2)$$
• **Example**
  – The impulse response of an LTI system is $h(n) = [2, 3, 0, 5]$
    • Find the difference equation representation
    • Draw the simulation diagram.