Department of Electrical Engineering
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ELEG 5173L Digital Signal Processing
Ch. 4 The Discrete Fourier Transform

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OUTLINE

- **The Discrete Fourier Transform (DFT)**
- **Properties**
- **Fast Fourier Transform**
- **Applications**

DISCRETE FOURIER TRANSFORM (DFT)

- **Review DTFT: Discrete-time Fourier transform**

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\Omega n} \qquad 0 \le \Omega \le \pi$$

  - Limitations for computer implementation:
    - 1. Infinite number of time domain samples $x(n)$ $-\infty \le n \le \infty$
      - Requires infinite memory
    - 2. Ω is a continuous variable
      - We can only approximate $X(\Omega)$ in a computer
  - Possible solutions:
    - 1. limit the number of time domain samples $x(n)$ $0 \le n \le N-1$

$$X_N(\Omega) = \sum_{n=0}^{N-1} x(n) e^{-j\Omega n} \qquad 0 \le \Omega \le \pi$$

    - 2. Sample $X_N(\Omega)$ in the frequency domain: $\Omega_k = \frac{2\pi k}{N}$ $0 \le k \le N-1$

$$X_N(\Omega_k) = \sum_{n=0}^{N-1} x(n) e^{-j\Omega_k n} = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N} n}$$

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DISCRETE FOURIER TRANSFORM (DFT)

- **Discrete Fourier Transform (DFT)**

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

  - Finite number time domain samples
  - Discrete frequency domain signal

$$x(n) \qquad 0 \le n \le N-1$$

$$X(k) \qquad 0 \le k \le N-1$$

- **Inverse Discrete Fourier Transform (IDFT)**

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}}$$

DFT

- **Periodicity in the frequency domain**
  - Recall: the DTFT is periodic $X(\Omega) = X(\Omega + 2\pi)$
  - The DFT $X(k)$ is periodic with period N

$$X(k) = X(k+N)$$

    - Proof

  - Time domain sampling leads to frequency domain repetition.

DFT

- **Periodicity in the time domain**
  - The time domain signal $x(n)$ from the IDFT is also periodic with period N

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi kn}{N}}$$

$$x(n)=x(n+N)$$

  - Proof

  - frequency domain sampling leads to time domain repetition.

DFT

- **Example**
  - Find the DTFT of $x(n) = (0.8)^n u(n),$
  - Find the DFT of $\hat{x}(n) = (0.8)^n u(n), \quad 0 \le n \le N-1$
  - Plot the DTFT, and plot the DFT when N = 5, 10, 20, and 50, respectively.

DFT

- **Example**
  - A finite-duration sequence of length L is given as follows. Find the N-point DFT of this sequence for $N \geq L$. Plot the frequency response.

$$x(n) = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

DFT

- **Relationship between DFT and DTFT**
  - DFT

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi kn}{N}} \qquad 0\le k\le N-1$$

  - DTFT

$$Y(\Omega)=\sum_{n=-\infty}^{+\infty}x(n)e^{-j\Omega n} \qquad 0\le \Omega\le \pi$$

  - Relationship

$$X(k)=Y(\Omega)\Big|_{\Omega=\frac{2\pi k}{N}}$$

    - DFT index k → angular digital frequency $2\pi\frac{k}{N}$ radians
    - DFT index k → angular analog frequency $2\pi\frac{k}{NT_s}$ radians/sec
    - $2\pi\frac{k}{N}\in[0,\frac{2\pi}{N},\cdots,2\pi\frac{N-1}{N}]$

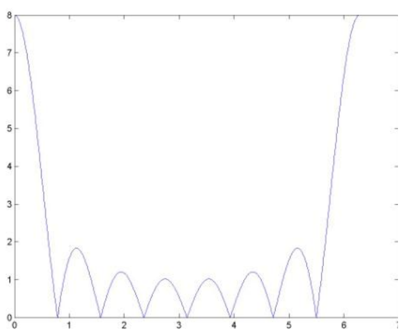
DFT

- **Frequency domain resolution**
  - $\Omega_k = 2\pi \frac{k}{N}$
  - Freq. domain resolution: Space between 2 freq. domain samples

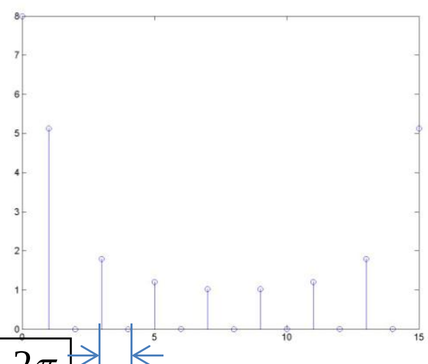
$$\Delta\Omega = \Omega_k - \Omega_{k-1} = \frac{2\pi}{N}$$

  - Larger N → smaller $\Delta\Omega = \frac{2\pi}{N}$ → better frequency domain resolution
  - Example:

$$x(n) = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & \text{otherwise} \end{cases}$$

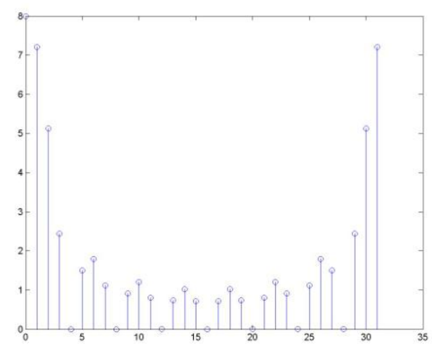


DTFT

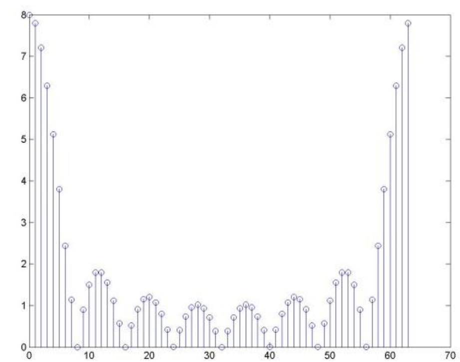


$$\boxed{\Delta\Omega = \frac{2\pi}{N}}$$

DFT N=16



DFT N=32



DFT N=64

DFT

- **Matrix representation of DFT**
  - DFT: let $W_N = e^{\frac{-j2\pi}{N}}$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

  - Define DFT matrix
    - The (k+1, n+1)-th element is W_N^{kn}

$$\mathbf{W} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

DFT

- **Matrix representation of DFT**

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

DFT

- **Matrix representation of IDFT**
  - DFT: let $W_N = e^{\frac{-j2\pi}{N}}$ $\qquad W_N^* = e^{\frac{j2\pi}{N}}$

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} X(n)(W_N^{kn})^*$$

  - Define IDFT matrix
    - The (k, n)-th element is $\left(W_N^{nk}\right)^* = W_N^{-nk}$

$$\mathbf{W}^H = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}$$

$\mathbf{W}^H$: the complex transpose of $\mathbf{W}$ (transpose the matrix, then take the complex conjugate of all the elements)

DFT

- **Matrix representation of IDFT**

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1}X(k)W_N^{-kn}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ x(N-1) \end{bmatrix} = \frac{1}{N}\begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\boxed{\mathbf{x}=\frac{1}{N}\mathbf{W}^H\mathbf{X}}$$

OUTLINE

- The Discrete Fourier Transform (DFT)
- **Properties**
- Fast Fourier Transform
- Applications

PROPERTIES

- **Linearity**
  - $x_1(n) \leftrightarrow X_1(k)$ $\qquad$ $x_2(n) \leftrightarrow X_2(k)$

$$ax_1(n) + bx_2(n) \leftrightarrow aX_1(k) + bX_2(k)$$

- **Periodicity**
  - If $x(n) \leftrightarrow X(k)$
  - Then

$$x(n) = x(n+N)$$

$$X(k) = X(k+N)$$

PROPERTIES

- **Circular shift**
  - circular shifting a length-N signal, $x(n)$, to the right by n_0 positions

$$x(n-n_0)_N$$

  - Example: N = 4

$$x(n)=[x(0),x(1),x(2),x(3)]$$
$$x(n-1)_4=[x(3),x(0),x(1),x(2)]$$
$$x(n-2)_4=[x(2),x(3),x(0),x(1)]$$
$$x(n-3)_4=[x(1),x(2),x(3),x(0)]$$

$(n-n_0)_N = n-n_0+pN$, where p is an integer chosen such that $0 \le n-n_0+pN \le N-1$

  - Why circular shift?
    - Recall: in DFT, $x(n)$ is periodic in N

$$x(n):\quad \cdots x(0),x(1),x(2),x(3),\,|\,x(0),x(1),x(2),x(3),\,|\,x(0),x(1),x(2),x(3)\cdots$$

$$x(n-1):\quad \cdots x(0),x(1),x(2),\,|\,x(3),x(0),x(1),x(2),\,|\,x(3),x(0),x(1),x(2),x(3)\cdots$$

PROPERTIES

- **Time shifting**
  - If $x(n) \leftrightarrow X(k)$
  - Then $x(n-n_0)_N \leftrightarrow X(k)\exp\left(-j\dfrac{2\pi}{N}kn_0\right)$
  - This is a circular shift
    - because $x(n)$ is periodic in N

PROPERTIES

- **Example**
  - Consider a sequence $\mathbf{x} = [1, -1, 2, 4, -2, 3]$
  - Find the DFT
  - If we circular shift x to the right by two locations, find the new sequence and its DFT

PROPERTIES

- **Circular convolution**
  - The circular convolution between two length-*N* sequences x(n) and h(n) is

$$y(n) = \sum_{k=0}^{N-1} x(k) h(n-k)_N$$

  - Graphical interpretation for *N* = 4

$$x(k) = [x(0), x(1), x(2), x(3)]$$

$$n = 0: \; h(-k)_N = [h(0), h(3), h(2), h(1)] \qquad y(0) = x(0)h(0) + x(1)h(3) + x(2)h(2) + x(3)h(1)$$

$$n = 1: \; h(1-k)_N = [h(1), h(0), h(3), h(2)] \qquad y(1) = x(0)h(1) + x(1)h(0) + x(2)h(3) + x(3)h(2)$$

$$n = 2: \; h(2-k)_N = [h(2), h(1), h(0), h(3)] \qquad y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) + x(3)h(3)$$

$$n = 3: \; h(3-k)_N = [h(3), h(2), h(1), h(0)] \qquad y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0)$$

PROPERTIES

- **Example**
  - Find the circular convolution of the following two sequences

$$x(n) = [2,1,2,1] \qquad h(n) = [1,2,3,4]$$

PROPERTIES

- **Example**
  - Find the circular convolution of the two sequences

$$x(n) = [3,1,2,5,4] \qquad h(n) = [1,2,3,4,5]$$

PROPERTIES

- **Circular convolution and DFT**
  - Consider two length-N sequences x(n) and h(n). There N-point DFTs are X(k) and H(k), respectively.

$$\sum_{m=0}^{N-1} x(m)h(n-m)_N \Leftrightarrow X(k)H(k)$$

  - Circular convolution in the time domain is equivalent to multiplication in the discrete frequency domain

PROPERTIES

- **Example**
  - Find the circular convolution of the two sequences by using DFT

$$x(n) = [2,1,2,1] \qquad\qquad h(n) = [1,2,3,4]$$

PROPERTIES

- **Multiplication of two sequences**
  - Consider two length-N sequences x(n) and h(n). Their N-point DFTs are X(k) and H(k), respectively.

$$x(n)h(n) \Leftrightarrow \frac{1}{N}\sum_{m=0}^{N-1} X(m)H(n-m)_N$$

  - Multiplication in the time domain is equivalent to circular convolution in the discrete frequency domain

PROPERTIES

- **Example**
  - Consider two length-N sequences. Find the circular convolution of their 4-point DFTs.

$$x(n) = [3,1,5,4] \qquad\qquad h(n) = [2,6,1,3]$$

PROPERTIES

- **Time-reversal**
  - If the N-point DFT of x(n) is X(k), then

$$x(-n)_N \Leftrightarrow X(-k)_N$$

  - Example
    - If the 6-point DFT of x(n) is X(k) = [3, -2, 4, -1, 5]. Find the DFT of $x(-n)_N$

PROPERTIES

- **Parseval's theorem**
  - If the N-point DFT of x(n) is X(k), then

$$\sum_{n=0}^{N-1}|x(n)|^2 \Leftrightarrow \frac{1}{N}\sum_{k=0}^{N-1}|X(k)|^2$$

  - Example
    - If $x(n)=[2+j,1,1-j,3]$, find $\sum_{k=0}^{3}|X(k)|^2$

OUTLINE

- The Discrete Fourier Transform (DFT)
- Properties
- **Fast Fourier Transform**
- Applications

FFT

- **Fast Fourier Transform (FFT)**
  - A faster implementation of DFT (NOT a new transform!)
  - The result is exactly the same as DFT, just the implementation is faster.
- **Complexity of DFT**

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

  - For each k, there are N complex multiplications
  - The above formula needs to be performed N times for $k = 0, 1, \ldots, N$-1
  - Total number of complex multiplications: N^2
  - Total number of complex multiplications for FFT: $N \log_2(N)$

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FFT

- **FFT**
  - There are many different ways of implementing FFT
    - Decimation in time
    - Decimation in frequency
    - …
  - It utilizes the following property

$$W_{N/2} = \exp\left(-j\frac{2\pi}{N/2}\right) = \exp\left(-j\frac{2\pi}{N}2\right) = \left[\exp\left(-j\frac{2\pi}{N}\right)\right]^2 = W_N^2$$

$$W_{N/2} = W_N^2$$

$$W_{N/2}^k = W_N^{2k}$$

FFT

- **FFT: decimation in time**

$$X(k) = \sum_{n\, even} x(n) W_N^{kn} + \sum_{n\, odd} x(n) W_N^{kn}$$

$$X(k) = \sum_{r=0}^{N/2-1} x(2r) W_N^{2rk} + \sum_{r=0}^{N/2-1} x(2r+1) W_N^{(2r+1)k}$$

let $\quad g(r) = x(2r) \qquad h(r) = x(2r+1)$

$$X(k) = \sum_{r=0}^{N/2-1} g(r) W_N^{2rk} + W_N^k \sum_{r=0}^{N/2-1} h(r) W_N^{2rk}$$

$$X(k) = \sum_{r=0}^{N/2-1} g(r) W_{N/2}^{rk} + W_N^k \sum_{r=0}^{N/2-1} h(r) W_{N/2}^{rk}$$

$$\boxed{X(k) = G(k) + W_N^k H(k)}$$

FFT

- **FFT: decimation in time (cont'd)**

$$X(k) = \sum_{r=0}^{N/2-1} g(r) W_{N/2}^{rk} + W_N^k \sum_{r=0}^{N/2-1} h(r) W_{N/2}^{rk}$$

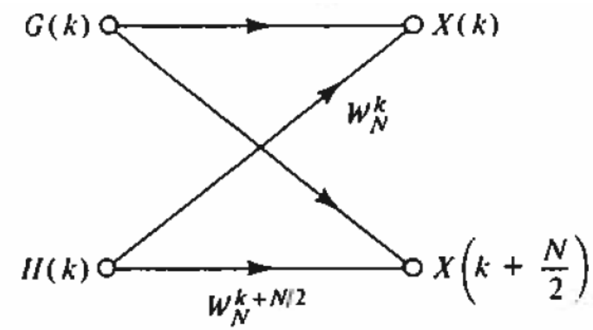
$$X(k) = G(k) + W_N^k H(k)$$

  - $G(k)$ is the $N/2$-point DFT of the even-indexed samples $g(r) = x(2r)$
  - $H(k)$ is the $N/2$-point DFT of the odd-indexed samples $h(r) = x(2r+1)$
  - Based on periodicity

$$G(k) = G(k + N/2) \qquad H(k) = H(k + N/2)$$

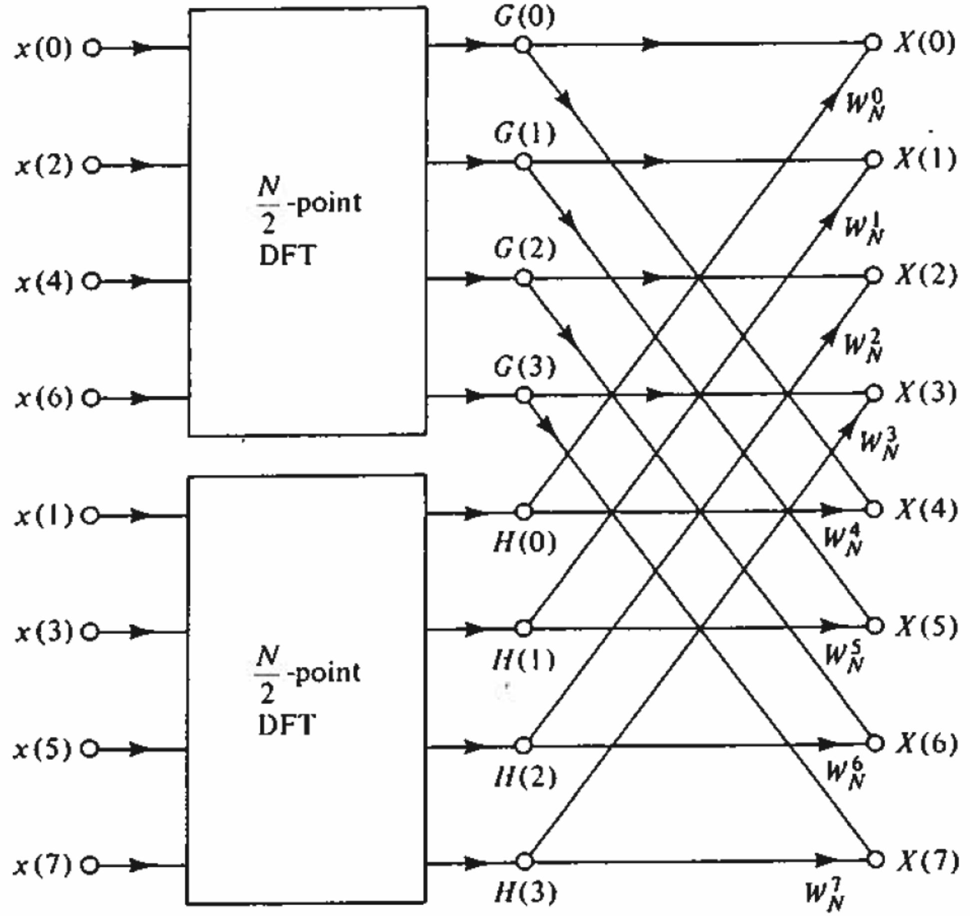
$$X(k + N/2) = G(k) + W_N^{k+N/2} H(k)$$

  - Butterfly



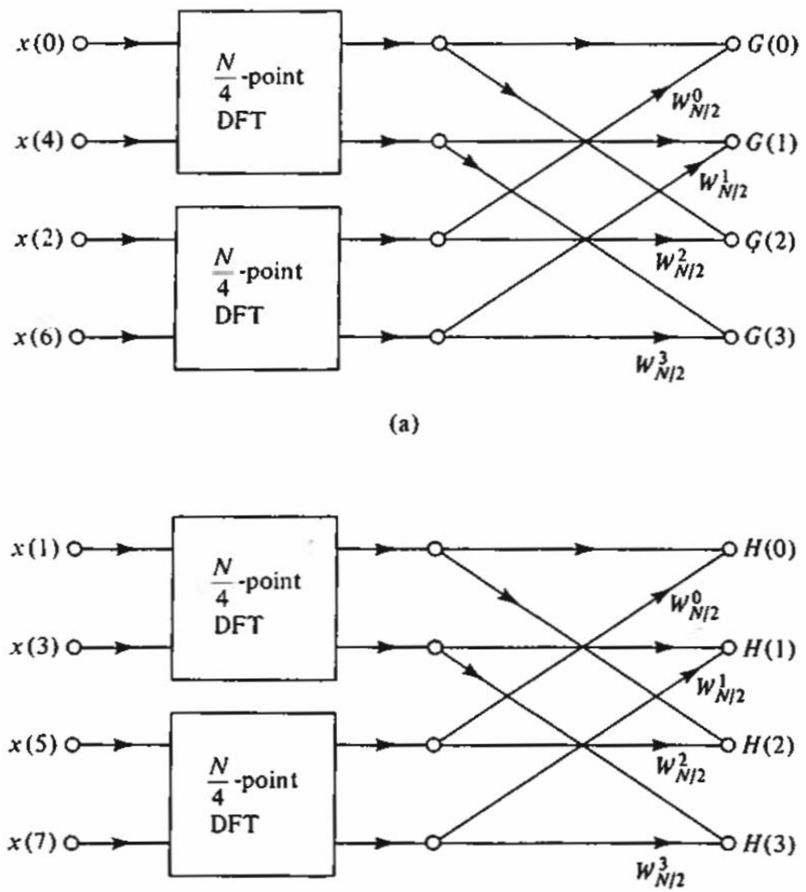
FFT

- **FFT: decimation in time**
  - The N-point DFT is decomposed into:
    - 2 N/2-point DFTs
    - N complex multiplications



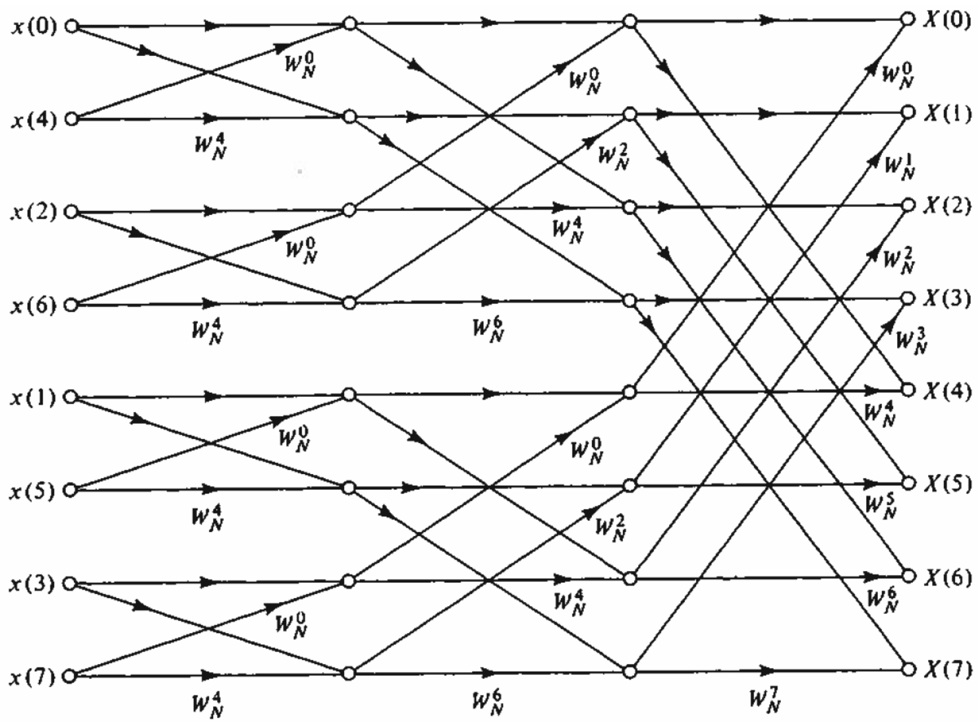
FFT

- **FFT: decimation in time**
  - Each N/2-point DFT can be decomposed into
    - 2 N/4-point DFTs
    - N/2 complex multiplications



FFT

- **FFT: Decimation in time**
  - Example 8-point FFT



  - Each stage requires *N* complex multiplications
  - There are $\log_2(N)$ stages
  - Total number of complex multiplications $N\log_2(N)$

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OUTLINE

- The Discrete Fourier Transform (DFT)
- Properties
- Fast Fourier Transform
- **Applications**

APPLICATIONS

- **Dual-tone multi-frequency (DTMF)**
  - In the touch tone telephone, pressing each key on the telephone will generate a DTMF signal
    - The signal contains two frequencies

	column 1 1209 Hz	column 2 1336 Hz	column 3 1477 Hz	column 4 1633 Hz
row 1 697 Hz	1	2	3	A
row 2 770 Hz	4	5	6	B
row 3 852 Hz	7	8	9	C
row 4 941 Hz	*	0	#	D

APPLICATIONS

- **DTMF**
  - Example
    - There is a DTMF signal with sampling frequency Fs = 8 KHz. The duration of the signal is 0.4 s. Performing FFT on the signal, and there are two peaks in the frequency domain at k = 308 and k = 484
    - How many samples are in the signal?
    - What is the resolution of the analog freuqency (in Hz)?
    - What are the analog frequencies (in Hz) corresponding to the two peaks?

APPLICATIONS

- **DTMF**
  - Example
    - There is a DTMF signal with sampling frequency Fs = 8 KHz. The duration of the signal is 0.5 s. The signal is generated by pressing the key '9'. Performing FFT on the signal. Which digital frequency indices (k) corresponding to the peaks in the frequency domain?