**Department of Electrical Engineering**
**University of Arkansas**

# ELEG 5173L Digital Signal Processing
# Ch. 3 Discrete-Time Fourier Transform

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# OUTLINE

- The Discrete-Time Fourier Transform (DTFT)
- Properties
- DTFT of Sampled Signals
- Upsampling and downsampling

# DTFT

- **Discrete-time Fourier Transform (DTFT)**

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\Omega n}$$

  - $\Omega$ (radians): digital frequency

- **Review: Z-transform:**

$$X(z) = \sum_{n=0}^{+\infty} x(n) z^{-n}$$

  - Replace z with $e^{j\Omega}$ . $X(\Omega) = X(z)\big|_{z=e^{j\Omega}}$

- **Review: Fourier transform:**

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}$$

  - $\omega$ (rads/sec): analog frequency

# DTFT

- **Relationship between DTFT and Fourier Transform**
  - Sample a continuous time signal $x_a(t)$ with a sampling period $T$

$$x_s(t) = x_a(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT) = \sum_{n=-\infty}^{+\infty}x_a(nT)\delta(t-nT)$$

  - The Fourier Transform of $y_s(t)$

$$X_s(\omega) = \int_{-\infty}^{+\infty}x_s(t)e^{-j\omega t}dt = \sum_{n=-\infty}^{+\infty}x_a(nT)e^{-j\omega nT}$$

  - Define: $\Omega = \omega T$
    - $\Omega$: digital frequency (unit: radians)
    - $\omega$: analog frequency (unit: radians/sec)

  - Let $x(n) = x_a(nT)$

$$X(\Omega) = X_s\left(\frac{\Omega}{T}\right)$$

# DTFT

- **Relationship between DTFT and Fourier Transform (Cont'd)**
  - The DTFT can be considered as the scaled version of the Fourier transform of the sampled continuous-time signal

$$X_s(\omega) = \int_{-\infty}^{+\infty} x_s(t) e^{-j\omega t} dt = \sum_{n=-\infty}^{+\infty} x_a(nT) e^{-j\omega nT}$$

$$\omega = \frac{\Omega}{T} \qquad\qquad x(n) = x_a(nT)$$

$$X(\Omega) = X_s\left(\frac{\Omega}{T}\right) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\Omega n}$$

# DTFT

- **Discrete Frequency** $\Omega$
  - Unit: radians (the unit of continuous frequency is radians/sec)
  - $X(\Omega)$ is a periodic function with period $2\pi$

$$X(\Omega+2\pi)=\sum_{n=-\infty}^{+\infty}x(n)e^{-j(\Omega+2\pi)n}=\sum_{n=-\infty}^{+\infty}x(n)e^{-j\Omega n}e^{-j2\pi n}=\sum_{n=-\infty}^{+\infty}x(n)e^{-j\Omega n}=X(\Omega)$$

  - We only need to consider $X(\Omega)$ for $-\pi\le\Omega\le\pi$
    - For Fourier transform, we need to consider $-\infty\le\omega\le\infty$
  - $\Omega=-\pi \rightarrow \omega=\dfrac{-\pi}{T} \rightarrow f=\dfrac{\omega}{2\pi}=-\dfrac{1}{2T}$
  - $\Omega=\pi \rightarrow \omega=\dfrac{\pi}{T} \rightarrow f=\dfrac{\omega}{2\pi}=\dfrac{1}{2T}$

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# DTFT

- **Example: find the DTFT of the following signal**
  - 1. $x(n) = \delta(n)$
  - 2. $x(n) = \alpha^n u(n), |\alpha| < 1$

# DTFT

- **Example**
  - Find the DTFT of $x(n) = u(n) - u(n-N)$

# DTFT

- **Example**
  - Find the DTFT of the following signal

$$x(n) = \alpha^{|n|}, |\alpha| < 1$$

- **Existence of DTFT**
  - The DTFT of $x(n)$ exists if $x(n)$ satisfies

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

# DTFT

- **Inverse DTFT**

$$x(n) = \frac{1}{2\pi}\int_{<2\pi>} X(\Omega)e^{j\Omega n}d\Omega$$

- **Example**
  - Find the inverse DTFT of $\delta(\Omega + \Omega_0)$

# DTFT

- **Example**
  - Find the DTFT of $\cos(\Omega_0 n + \theta)$

# OUTLINE

- The Discrete-Time Fourier Transform (DTFT)
- **Properties**
- DTFT of Sampled Signals
- Upsampling and downsampling

# PROPERTIES

- **Periodicity**

$$X(\Omega + 2\pi) = X(\Omega)$$

- **Linearity**
  - If $x_1(n) \leftrightarrow X_1(\Omega)$ $x_2(n) \leftrightarrow X_2(\Omega)$
  - Then

$$\alpha x_1(n) + \beta x_2(n) \leftrightarrow \alpha X_1(\Omega) + \beta X_2(\Omega)$$

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# PROPERTIES

- **Time shifting**
  - If $x(n) \leftrightarrow X(\Omega)$
  - Then

$$x(n-n_0) \leftrightarrow X(\Omega)e^{-j\Omega n_0}$$

- **Frequency shifting**
  - If $x(n) \leftrightarrow X(\Omega)$
  - Then

$$e^{j\Omega_0 n}x(n) \leftrightarrow X(\Omega-\Omega_0)$$

# PROPERTIES

- **Differentiation in Frequency**
  - If $x(n) \leftrightarrow X(\Omega)$
  - Then $$nx(n) \leftrightarrow j\frac{dX(\Omega)}{d\Omega}$$
  - Review $nx(n) \leftrightarrow -z\frac{d}{dz}X(z)$
- **Example**
  - Find the DTFT of $n\alpha^n u(n)$ $|\alpha| < 1$

# PROPERTIES

- **Convolution**
  - If $$y(n) = h(n) \otimes x(n)$$
  - Then
$$Y(\Omega) = H(\Omega)X(\Omega)$$

- **Example**
  - Find the frequency response of the system $y(n) = x(n - n_0)$

# PROPERTIES

- **Example**
  - A LTI system with impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$

    if the input is $x(n) = \left(\frac{1}{3}\right)^n u(n)$

    Find the output

# PROPERTIES

- **Example**
  - The frequency response of an ideal low pass filter is

$$H(\Omega)=\begin{cases}1, & 0\le|\Omega|\le\Omega_c\\ 0, & \Omega_c<|\Omega|<\pi\end{cases}$$

find the impulse response.

(1) Assume $\Omega_c = 0.5\pi$. If the sampling rate of the input signal is 1 KHz, what is the cutoff frequency for the analog signal?

(2) If the sampling rate of the input signal is 2 KHz, what is the cutoff frequency of the analog signal?

# PROPERTIES

- **Modulation**
  - If $y(n) = x_1(n)x_2(n)$
  - Then

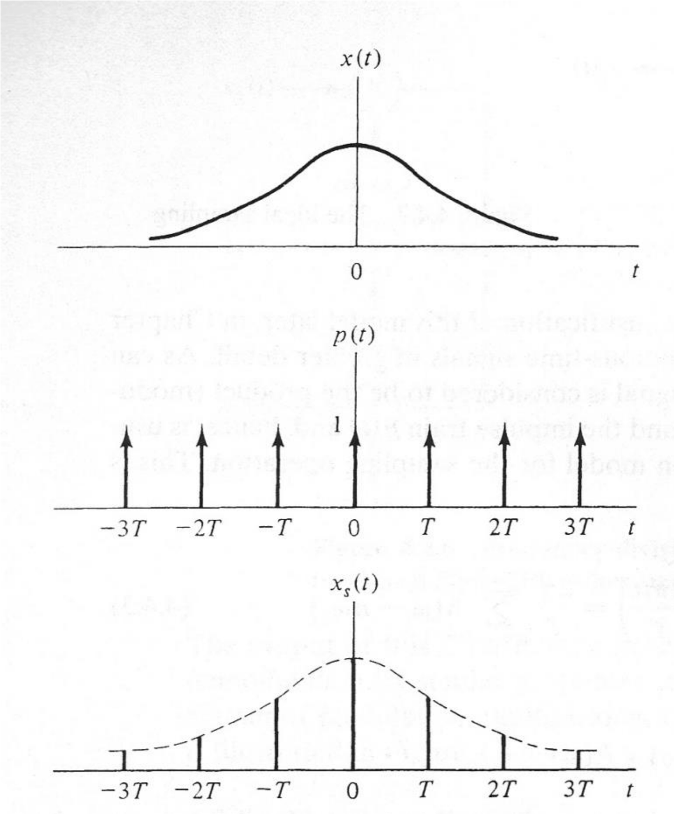
$$Y(\Omega) = \frac{1}{2\pi}\int_0^{2\pi} X_1(p)X_2(\Omega - p)dp$$

# OUTLINE

- The Discrete-Time Fourier Transform (DTFT)
- Properties
- **Application: DTFT of Sampled Signals**
- Upsampling and downsampling

# APPLICATION: SAMPLING THEOREM

- **Sampling theorem: time domain**
  - Sampling: convert the continuous-time signal to discrete-time signal.




$$x_a(t)$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

sampling period

$$x_s(t) = x_a(t)\,p(t)$$

# APPLICATION: SAMPLING THEOREM

- **Sampling theorem: frequency domain**
  - Fourier transform of the impulse train
    - impulse train is periodic

Fourier series

$$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)=\frac{1}{T}\sum_{n=-\infty}^{+\infty}1\times e^{jn\omega_s t}\qquad \omega_s=\frac{2\pi}{T}$$

- Find Fourier transform on both sides

$$P(\omega)=\frac{2\pi}{T}\sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_s)$$

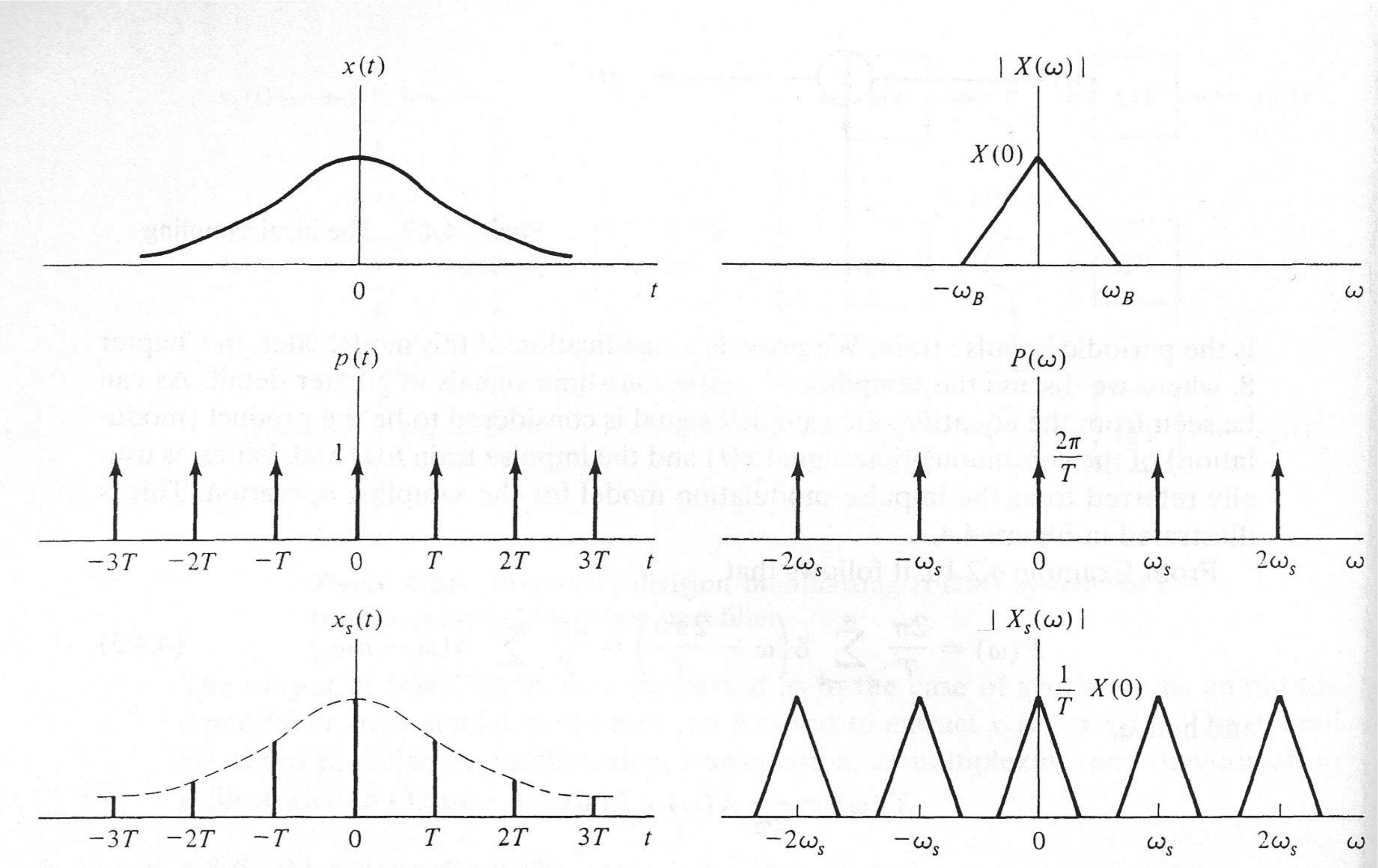
- Time domain multiplication ➔ Frequency domain convolution

$$x(t)p(t)\Leftrightarrow\frac{1}{2\pi}\left[X_a(\omega)\otimes P(\omega)\right]$$

$$x_s(t)=x(t)p(t)\Leftrightarrow X_s(\omega)=\frac{1}{T}\sum_{n=-\infty}^{+\infty}X_a\left(\omega-n\frac{2\pi}{T}\right)$$

# APPLICATION: SAMPLING THEOREM

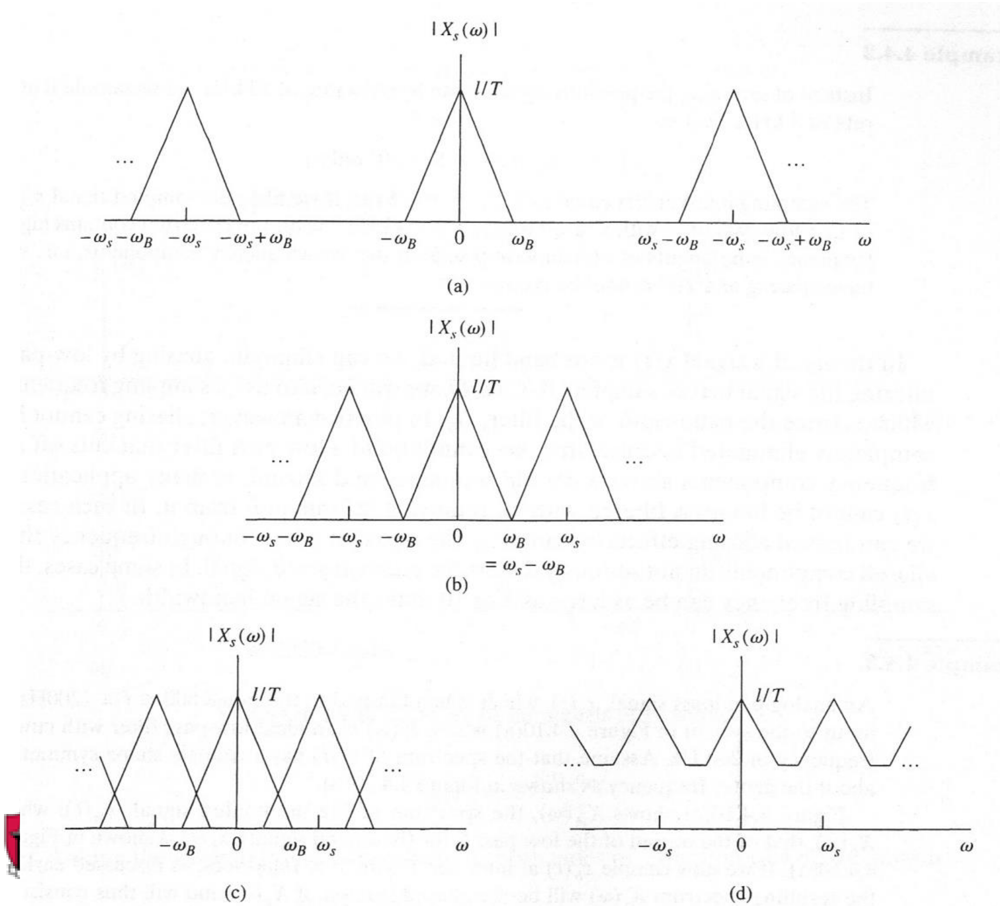
- **Sampling theorem: frequency domain**
  - Sampling in time domain ➔ Repetition in frequency domain



# APPLICATION: SAMPLING THEOREM

- **Sampling theorem**
  - If the sampling rate is twice of the bandwidth, then the original signal can be perfectly reconstructed from the samples.

$$\omega_s > 2\omega_B$$




$$\omega_s > 2\omega_B$$

$$\omega_s = 2\omega_B$$

$$\omega_s < 2\omega_B$$

aliasing

# APPLICATION: SAMPLING THEOREM

- **Analog signal $x_a(t)$ v.s. sampled signal $x_s(t)$ v.s. discrete-time $x(n)$ signal**

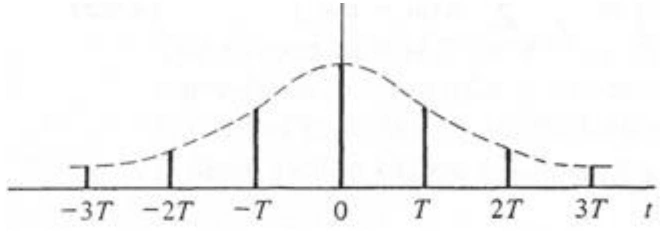
$$x(n) = x_a(nT)$$

Discrete-time signal

Samples of continuous-time signal

$$x_s(t) = x_a(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT) = \sum_{n=-\infty}^{+\infty}x_a(nT)\delta(t-nT) = \sum_{n=-\infty}^{+\infty}x(n)\delta(t-nT)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty}x(n)\delta(t-nT)$$



# APPLICATION: SAMPLING THEOREM

- **Fourier transform of sampled signal**

$$X_s(\omega) = \int_{-\infty}^{+\infty} x_s(t) e^{-j\omega t} dt = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega nT}$$

- **DTFT of discrete-time signal**

$$X(\Omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\Omega n}$$

- **Relationship**

$$X(\Omega) = X_s\left(\frac{\Omega}{T}\right)$$

DTFT of discrete-time signal

Fourier transform of continuous-time signal

# APPLICATION: SAMPLING THEOREM

- **Relationship among** $X(\Omega), X_s(\omega), X_a(\omega)$

$$X(\Omega) = X_s\left(\frac{\Omega}{T}\right)$$

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} X_a\left(\omega - n\frac{2\pi}{T}\right)$$

$$X(\Omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} X_a\left(\frac{\Omega}{T} - n\frac{2\pi}{T}\right)$$

$$\omega = \frac{\Omega}{T}$$

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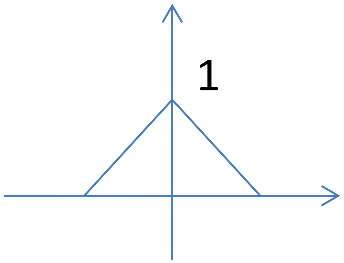
# APPLICATION: SAMPLING THEOREM

- **Example**
  - Consider the analog signal $x_a(t)$ with Fourier transform shown in the figure. The signal has a one-sided bandwidth 5 Hz. The signal is sampled with a sampling period $T = 25ms$

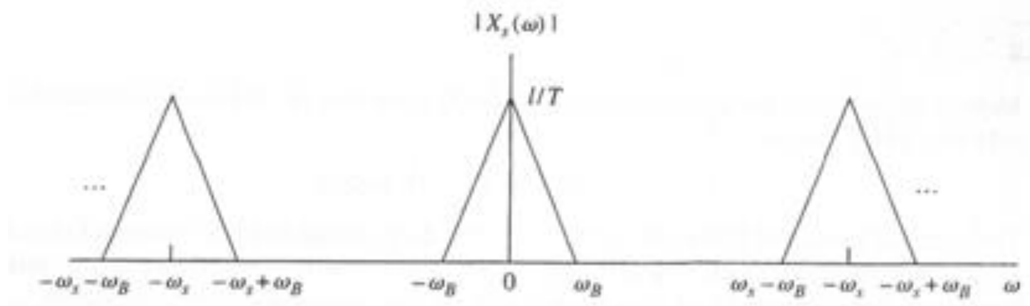
    Draw $X_s(\omega)$ and $X(\Omega)$



# APPLICATION: SAMPLING THEOREM

- **Reconstruction of sampled signal**
  - If there is no aliasing during sampling, the analog signal can be reconstructed by passing the sampled signal through an ideal low pass filter with cutoff frequency $\omega_B$

$$H(\omega) = \begin{cases} T, & |\omega| < \dfrac{\omega_s}{2} \\ 0, & \text{otherwise} \end{cases}$$




$$h(t) = \operatorname{sin}c\left(\frac{t}{T}\right)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x(n)\delta(t - nT)$$

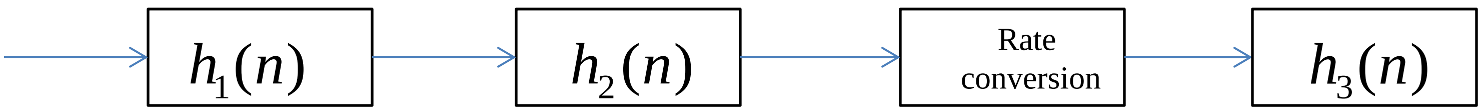
$$\hat{x}_a(t) = x_s(t) \otimes h(t) = \sum_{n=-\infty}^{+\infty} x(n)\operatorname{sin}c\left(\frac{t - nT}{T}\right)$$

# OUTLINE

- **The Discrete-Time Fourier Transform (DTFT)**
- **Properties**
- **Application: DTFT of Sampled Signals**
- **Upsampling and downsampling**

# UPSAMPLING AND DOWNSAMPLING

- **Sampling rate conversion**
  - In many applications, we may have to change the sampling rate of a signal as the signal undergoes successive stage of processing.
    - E.g.




Sampling period $T_1$ $T_1$ $T_2$

  - If $h_2(n)$ is a low pass filter with the cut-off frequency being $\Omega_0 = \frac{\pi}{u}$ then what is the largest value of $T_2$ ?

# UPSAMPLING AND DOWNSAMPLING

- **Sampling rate conversion**
  - Sampling rate conversion can be directly performed in the digital domain
  - Downsampling (decimation)
    - Reduce the sampling frequency (less samples per unit time)
  - Upsampling (interpolation)
    - Increase the sampling frequency (more samples per unit time)

# UPSAMPLING AND DOWNSAMPLING

- **Downsampling (decimation)**
  - Assume a digital signal, $x(n)$ , is obtained by sampling an analog signal $x_a(t)$ at a sampling period $T$ (or a sampling frequency $1/T$).

$$x(n) = x_a(nT)$$

  - The signal can also be sampled with a sampling period $MT$ (or a lower sampling frequency $1/(MT)$ )

$$x_d(n) = x_a(nMT)$$

  - Relationship between $x(n)$ and $x_d(n)$

$$x_d(n) = x(nM)$$

  - $x_d(n)$ is the downsampled (or decimated) version of $x(n)$
    - $x_d(n)$ can be obtained by picking one sample out of every $M$ consecutive samples of $x(n)$ → decimation.

$$x(n) = [x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7), x(8), \cdots]$$

$M = 3$ $\quad x_d(n) = [x(0), \quad x(3), \quad x(6), \quad , \cdots]$

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# UPSAMPLING AND DOWNSAMPLING

- **Downsampling (decimation): frequency domain**
  - Assume all the sampling rates are higher than the Nyquist sampling rate
  - The DTFT of $x(n)$ with a sampling period $T$

$$X(\Omega) = \frac{1}{T} X_a\left(\frac{\Omega}{T}\right) \qquad -\pi \le \Omega \le \pi$$

  - The DTFT of $x_d(n)$ with a sampling period MT

$$X_d(\Omega) = \frac{1}{MT} X_a\left(\frac{\Omega}{MT}\right) \qquad -\pi \le \Omega \le \pi$$

$$X_d(\Omega) = \frac{1}{M} X\left(\frac{\Omega}{M}\right)$$

# UPSAMPLING AND DOWNSAMPLING

- **Example**
  - Consider the frequency response of an ideal low pass filter

$$H(\Omega) = \begin{cases} 1, & -\dfrac{\pi}{2} \le \Omega \le \dfrac{\pi}{2} \\ 0, & -\pi \le \Omega \le -\dfrac{\pi}{2}, \dfrac{\pi}{2} \le \Omega \le \pi \end{cases}$$

    - Determine $h(n)$
    - If we downsample h(n) with a factor of M = 2. Find the downsampled response in both the time and frequency domains.

# UPSAMPLING AND DOWNSAMPLING

- **Upsampling (interpolation)**
  - Assume a digital signal, $x(n)$, is obtained by sampling an analog signal $x_a(t)$ at a sampling period $T$ (or a sampling frequency $1/T$).

$$x(n) = x_a(nT)$$

  - Upsampling by a factor of $L$
    - Inserting $L$-1 zeros between the original samples

$$x(n) = [x(0), \quad x(1), \quad x(2), \quad x(3), \cdots]$$

$L = 3$

$$x_u(n) = [x(0),0,0, x(1),0,0, x(2),0,0, x(3),\cdots]$$

  - The effective sampling period of the upsampled signal $\frac{T}{L}$ (sampling rate: $\frac{L}{T}$ )

$$x_u(n) = x_u(kL) = x(k) \qquad \text{if } \; n/L = k \text{ is an integer}$$

$$x_u(n) = 0, \qquad \text{if } \; n/L \neq \text{integer}$$

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# UPSAMPLING AND DOWNSAMPLING

- **Upsampling (interpolation): frequency domain**
  - The DTFT of $x_u(n)$ with a sampling period *T/L*

$$X_u(\Omega) = \sum_{n=-\infty}^{+\infty} x_u(n) e^{-j\Omega n} = \sum_{k=-\infty}^{+\infty} x(k) e^{-j\Omega k L} = X(L\Omega)$$

- The DTFT of $X(\Omega)$ is compressed in the frequency domain

# UPSAMPLING AND DOWNSAMPLING

- **Example**
  - A discrete pulse is given by $x(n) = u(n)-u(n-4)$. Suppose we upsample $x(n)$ by a factor of $L = 3$. Find the DTFT of the original and upsampled signals.