**Department of Electrical Engineering**
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# ELEG 5173L Digital Signal Processing
# Ch. 2 The Z-Transform

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# OUTLINE

- **The Z-Transform**
- **Properties**
- **Inverse Z-Transform**
- **Z-Transform of LTI system**

# Z-TRANSFORM

- **Bilateral Z-transform**

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

- **Unilateral Z-transform**

$$X(z) = \sum_{n=0}^{+\infty} x(n) z^{-n}$$

- **Z-transform:**
  - Can simplify the analysis of discrete-time LTI systems
    - Analyze the system in z-domain instead of time domain
  - Doesn't have any physical meaning (the frequency domain representation of discrete-time signal can be obtained through discrete-time Fourier transform)
  - Counterpart for continuous-time systems: Laplace transform.

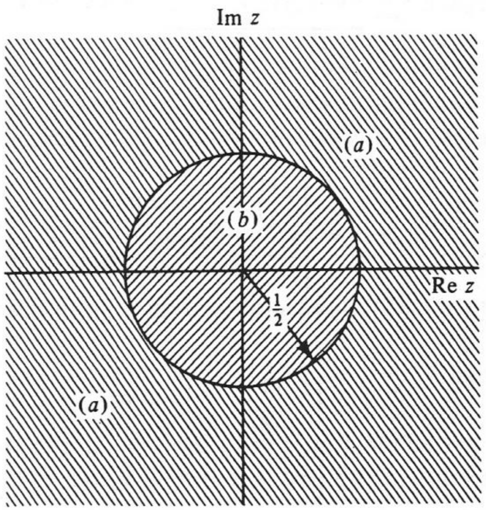
# Z-TRANSFORM

- **Example: find Z-transforms**
  - 1. $x(n) = \delta(n)$
  - 2. $x(n) = \left(\frac{1}{2}\right)^n u(n)$

# Z-TRANSFORM

- **Example**
  - 3. $x(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$

- **Region of convergence (ROC)**



# Z-TRANSFORM: CONVERGENCE

- **ROC of causal signal**

$$x(n) = \alpha^n u(n)$$

- **ROC of anti-causal signal**

$$x(n) = \beta^n u(-n-1)$$

# Z-TRANSFORM

- **Example: find the Z-transforms for the following signals**

$$x(n) = \left(\frac{1}{2}\right)^n u(-n) + \left(\frac{1}{3}\right)^n u(n)$$

# Z-TRANSFORM

- **Example: find the Z-transforms for the following signals**

$$x(n) = \begin{cases} 3^n, & n < 2 \\ \left(\frac{1}{3}\right)^n, & n \geq 2 \end{cases}$$

# Z-TRANSFORM: TRANSFORM TABLE

| | | |
|---|---|---|
| 1. $\delta(n)$ | $1$ | 0 |
| 2. $\delta(n-m)$ | $z^{-m}$ | 0 |
| 3. $u(n)$ | $\dfrac{z}{z-1}$ | 1 |
| 4. $n$ | $\dfrac{z}{(z-1)^2}$ | 1 |
| 5. $n^2$ | $\dfrac{z(z+1)}{(z-1)^3}$ | 1 |
| 6. $a^n$ | $\dfrac{z}{z-a}$ | $\lvert a \rvert$ |
| 7. $na^n$ | $\dfrac{az}{(z-a)^2}$ | $\lvert a \rvert$ |
| 8. $(n+1)a^n$ | $\dfrac{z^2}{(z-a)^2}$ | $\lvert a \rvert$ |
| 9. $\dfrac{(n+1)(n+2)\cdots(n+m)a^n}{m!}$ | $\dfrac{z^{m+1}}{(z-a)^{m+1}}$ | $\lvert a \rvert$ |
| 10. $\cos\Omega_0 n$ | $\dfrac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$ | 1 |

# Z-TRANSFORM: TRANSFORM TABLE

| | | |
|---|---|---|
| 11. $\sin \Omega_0 n$ | $\dfrac{z \sin \Omega_0}{z^2 - 2z \cos \Omega_0 + 1}$ | 1 |
| 12. $a^n \cos \Omega_0 n$ | $\dfrac{z(z - a \cos \Omega_0)}{z^2 - 2za \cos \Omega_0 + a^2}$ | $\lvert a \rvert$ |
| 13. $a^n \sin \Omega_0 n$ | $\dfrac{za \sin \Omega_0}{z^2 - 2za \cos \Omega_0 + a^2}$ | $\lvert a \rvert$ |
| 14. $\exp[-anT]$ | $\dfrac{z}{z - \exp[-aT]}$ | $\lvert \exp[-aT] \rvert$ |
| 15. $nT$ | $\dfrac{Tz}{(z-1)^2}$ | 1 |
| 16. $nT \exp[-anT]$ | $\dfrac{Tz \exp[-aT]}{[z - \exp[-aT]]^2}$ | $\lvert \exp[-aT] \rvert$ |
| 17. $\cos n\omega_0 T$ | $\dfrac{z(z - \cos \omega_0 T)}{z^2 - 2z \cos \omega_0 T + 1}$ | 1 |
| 18. $\sin n\omega_0 T$ | $\dfrac{z \sin \omega_0 T}{z^2 - 2z \cos \omega_0 T + 1}$ | 1 |
| 19. $\exp[-anT] \cos n\omega_0 T$ | $\dfrac{z[z - \exp[-aT] \cos \omega_0 T]}{z^2 - 2z \exp[-aT] \cos \omega_0 T + \exp[-2aT]}$ | $\lvert \exp[-aT] \rvert$ |
| 20. $\exp[-anT] \sin n\omega_0 T$ | $\dfrac{z[z - \exp[-aT] \sin \omega_0 T]}{z^2 - 2z \exp[-aT] \cos \omega_0 T + \exp[-2aT]}$ | $\lvert \exp[-aT] \rvert$ |

# OUTLINE

- The Z-Transform
- **Properties**
- Inverse Z-Transform
- Z-Transform of LTI system

# PROPERTIES

- **Linearity**
  - If

$$Z[x_1(n)] = X_1(z) \qquad Z[x_2(n)] = X_2(z)$$

  - Then

$$Z[a_1x_1(n) + a_2x_2(n)] = a_1X_1(z) + a_2X_2(z)$$

# PROPERTIES

- **Time Shifting**
  - Let $x(n)$ be a causal sequence with the Z-transform $X(z)$
  - Then

$$Z\left[x(n+n_0)\right] = z^{n_0} X(z) - z^{n_0} \sum_{m=0}^{n_0-1} x(m) z^{-m}$$

$$Z\left[x(n-n_0)\right] = z^{-n_0} X(z) + z^{-n_0} \sum_{m=-n_0}^{-1} x(m) z^{-m}$$

# PROPERTIES

- **Example**
  - Solve the difference equation with initial condition $y(-1) = 3$

$$y(n) - \frac{1}{2} y(n-1) = \delta(n)$$

# PROPERTIES

- **Example**
  - Solve the difference equation with initial condition $y(1)=1$ $y(0)=1$

$$y(n+2)-y(n+1)+\frac{2}{9}y(n)=u(n)$$

# PROPERTIES

- **Frequency scaling**
  - If

$$Z[x(n)] = X(z)$$

  - Then

$$Z[a^n x(n)] = X(a^{-1}z)$$

# PROPERTIES

- **Example**
  - Find the Z-transform of $x(n) = a^n \cos(\omega n) u(n)$

# PROPERTIES

- **Differentiation with respect to z**
  - If

$$Z[x(n)] = X(z)$$

  - Then

$$Z[nx(n)] = -z\frac{d}{dz}X(z)$$

# PROPERTIES

- **Example**
  - Find the Z-transform of $y(n) = n(n+1)u(n)$

# PROPERTIES

- **Initial value**

$$\lim_{z \to \infty} X(z) = x(0)$$

- **Final value**

$$\lim_{z \to 1} (1 - z^{-1}) X(z) = x(\infty)$$

# PROPERTIES

- **Example**
  - Find the initial value and final value of the following signal.

$$X(z) = \frac{z^2 - 2z + 3}{(z-1)(z-0.5)(z^2+z+1)}$$

# PROPERTIES

- **Convolution**
  - If
    $$Z[x(n)] = X(z)$$
    $$Z[h(n)] = H(z)$$
  - Then
    $$Z[x(n) \otimes h(n)] = X(z)H(z)$$
- **Example**
  - Find the convolution of the following two sequences

$$x(n) = [1,2,0,-1] \qquad y(n) = [1,3,-1]$$

# PROPERTIES

| | | |
|---|---|---|
| 1. Linearity | $a_1x_1(n) + a_2x_2(n)$ | $a_1X_1(z) + a_2X_2(z)$ |
| 2. Time shift | $x(n + n_0)$ | $z^{n_0}\left[X(z) - \sum_{m=0}^{n_0-1} x(m)z^{-m}\right]$ |
| | $x(n - n_0)$ | $z^{-n_0}\left[X(z) + \sum_{m=-n_0}^{-1} x(m)z^{-m}\right]$ |
| 3. Frequency scaling | $a^n x(n)$ | $X(a^{-1}z)$ |
| 4. Multiplication by $n$ | $nx(n)$ | $-z\frac{d}{dz}X(z)$ |
| | $n^k x(n)$ | $\left(-z\frac{d}{dz}\right)^k X(z)$ |
| 5. Convolution | $x_1(n) * x_2(n)$ | $X_1(z)X_2(z)$ |

# OUTLINE

- **The Z-Transform**
- **Properties**
- **Inverse Z-Transform**
- **Z-Transform of LTI system**

# INVERSE Z-TRANSFORM

- **Review**

$$a^n u(n) \leftrightarrow \frac{z}{z-a} \qquad\qquad na^n u(n) \leftrightarrow \frac{az}{(z-a)^2}$$

- **Inverse Z-Transform by partial fraction expansion**
  - Expand X(z) in the form of $\frac{z}{z-a}$ , $\frac{az}{(z-a)^2}$ , etc.

$$X(z) = \frac{1}{(z-a_1)(z-a_2)} \qquad \frac{1}{z}X(z) = \frac{1}{z(z-a_1)(z-a_2)} = \frac{A_0}{z} + \frac{A_1}{z-a_1} + \frac{A_2}{z-a_2}$$

$$A_0 = z\frac{X(z)}{z}\bigg|_{z=0} \qquad A_1 = (z-a_1)\frac{X(z)}{z}\bigg|_{z=a_1} \qquad A_2 = (z-a_2)\frac{X(z)}{z}\bigg|_{z=a_2}$$

$$X(z) = \frac{1}{z(z-a_1)(z-a_2)} = A_0 + A_1\frac{z}{z-a_1} + A_2\frac{z}{z-a_2}$$

# INVERSE Z-TRANSFORM

- **Example**
  - Find the inverse Z-Transform of

$$X(z) = \frac{1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} \qquad |z| > \frac{1}{2}$$

# INVERSE Z-TRANSFORM

- **Solve the difference equation**

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = \delta(n)$$

# INVERSE Z-TRANSFORM

- **Example**
  - Find the convolution $a^n u(n) \otimes b^n u(n)$

# INVERSE Z-TRANSFORM

- **Example**
  - Find the following convolutions

$$a^n u(n) \otimes \delta(n-1)$$

$$a^n u(n) \otimes \delta(n+1)$$

# OUTLINE

- The Z-Transform
- Properties
- Inverse Z-Transform
- **Z-Transform of LTI system**

# LTI SYSTEM

- **Transfer function of discrete-time LTI system**
  - ad

$$y(n) = x(n) \otimes h(n)$$

$$Y(z) = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

- **Transfer function of discrete-time LTI system**

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

  - Z-transform on both sides:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

# LTI SYSTEM

- **Example**
  - Let the step response of a LTI system be as follows. Find the transfer function

$$y(n) = \frac{6}{5}u(n) - \frac{1}{2}\left(\frac{1}{2}\right)^n u(n) + \frac{2}{15}\left(-\frac{1}{4}\right)^n u(n)$$

# LTI SYSTEM

- **Zeros and poles**

$$H(z) = \frac{(z - z_M)(z - z_{M-1})\cdots(z - z_1)}{(z - p_N)(z - p_{N-1})\cdots(z - p_1)}$$

  - Zeros: $z_1, z_2, \cdots, z_M$
  - Poles: $p_1, p_2, \cdots, p_N$

- **Stability**
  - A discrete-time LTI system is stable if all the poles are inside the unit circle.
  - A discrete-time LTI system is unstable if at least one pole is on or outside the unit circle.

  - Review: a continuous-time LTI system is stable if all the poles are on the left half plane.

# LTI SYSTEM

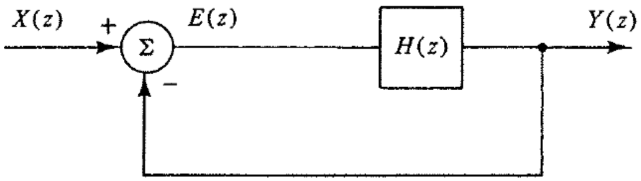
- **Example**
  - Consider a LTI system described by the difference equation. Find the transfer function and the zeros and poles. Is the system stable?

$$y(n) - 2y(n-1) + 2y(n-2) = x(n) + \frac{1}{2}x(n-1)$$

# LTI SYSTEM

- **Example**
  - Find the transfer function of the system shown in the following diagram. If k = 1, is the system stable?

$$H(z) = \frac{0.8z}{(z-0.5)(z-0.8)}$$



# LTI SYSTEM

- **Matlab**
  - Example

$$H(z) = \frac{2 + z^{-2}}{1 + 3z^{-1} + 2z^{-2}}$$

```
% numerator coefficients
b = [2, 0, 1];
% denominator coefficients
a = [1, 3, 2];

[r, p, k] = residuez(b, a)
```

r = [4.5, -3], p =[-2, -1], k = 0.5

$$H(z) = \frac{r_1}{1 - p_1 z^{-1}} + \frac{r_2}{1 - p_2 z^{-1}} + k = \frac{4.5}{1 - (-2)z^{-1}} + \frac{-3}{1 - (-1)z^{-1}} + 0.5$$

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# LTI SYSTEM

- **Matlab**
  - Example (Cont'd) $$H(z) = \frac{2 + z^{-2}}{1 + 3z^{-1} + 2z^{-2}}$$

```
% numerator coeffcients
b = [2, 0, 1];
% denominator coefficients
a = [1, 3, 2];

 % partial fraction expansion
[r, p, k] = residuez(b, a)

% find the zeros
z = roots(b);

% plot the poles and zeros
zplane(b, a);

% find the output to the system with an input x
x = [2, 1, -2, 3];
y = filter(b, a, x);
```

# LTI SYSTEM

- **Matlab**
  - Example (multiple poles) $$H(z) = \frac{2 + z^{-2}}{1 + 4z^{-1} + 4z^{-2}}$$

```
% numerator coeffcients
b = [2, 0, 1];
% denominator coefficients
a = [1, 4, 4];

 % partial fraction expansion
[r, p, k] = residuez(b, a)
```

r = [-0.5, 2.25], p =[-2, -2], k = 0

$$H(z) = \frac{r_1}{1 - p_1 z^{-1}} + \frac{r_2}{(1 - p_2 z^{-1})^2} = \frac{-0.5}{1 - (-2)z^{-1}} + \frac{2.25}{[1 - (-2)z^{-1}]^2}$$

$$na^n u(n) \leftrightarrow \frac{az}{(z-a)^2}$$

$$h(n) = -0.5(-2)^n u(n) + 2.25(n+1)(-2)^{n+1} u(n+1)$$