Department of Electrical Engineering
University of Arkansas

ELEG 4603/5173L Digital Signal Processing
Ch. 1 Discrete-Time Signals and Systems

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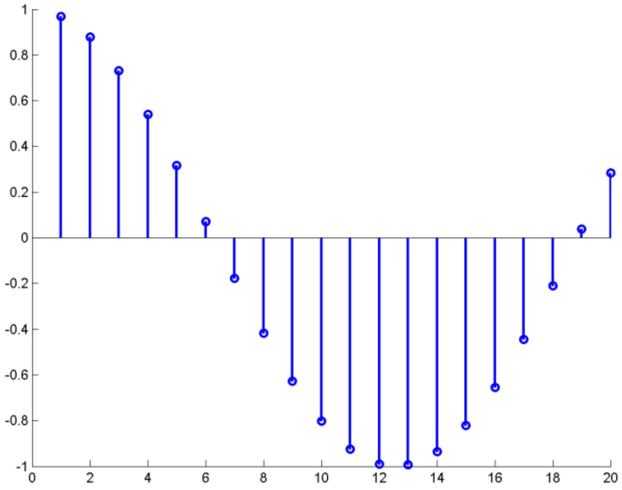
OUTLINE

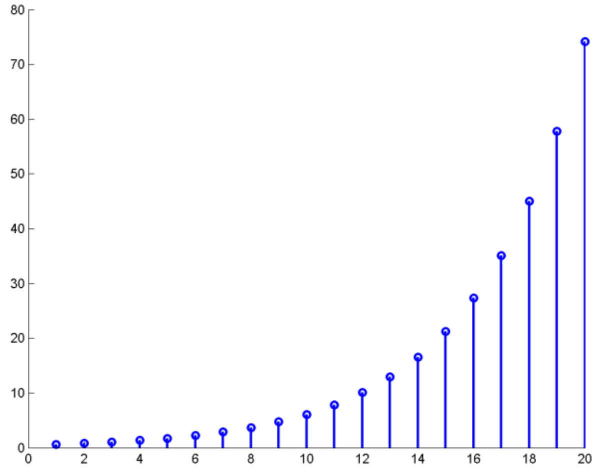
- **Classifications of discrete-time signals**
- **Elementary discrete-time signals**
- **Linear time-invariant (LTI) discrete-time systems**
- **Causality and stability**
- **Difference equation representation of LTI systems**

SIGNAL CLASSICIATION

- **Discrete-time signal**
  - A signal that is defined only at discrete instants of time.
  - Represented as $x(n), \quad n = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$



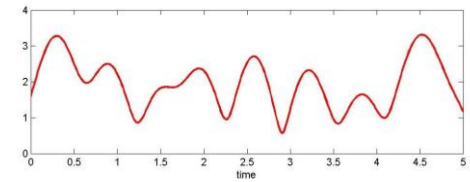
$$x(n) = \cos\left(\frac{n}{4}\right)$$



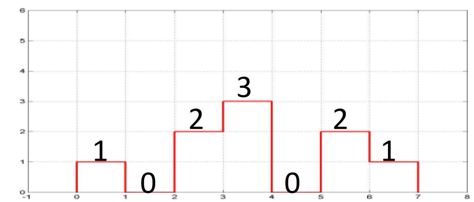
$$x(n) = \frac{1}{2}\exp\left(\frac{n}{4}\right)$$

SIGNAL CLASSICIATION

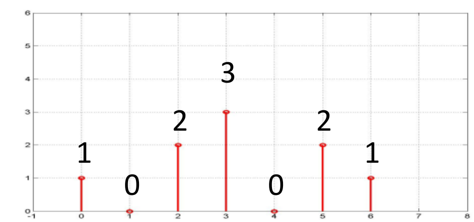
- **Review: Analog v.s. digital**
  - Continuous-time signal $x(t)$,
    - continuous-time, continuous amplitude→ analog signal
      - Example: speech signal



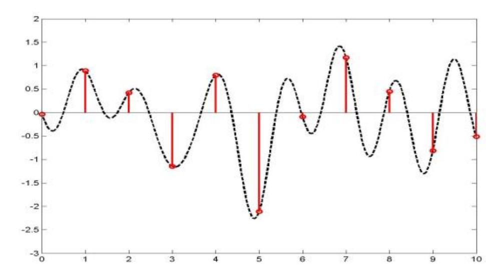
    - Continuous-time, discrete amplitude
      - Example: traffic light



  - Discrete-time signal $x(n)$,
    - Discrete-time, discrete-amplitude → digital signal
      - Example: Telegraph, text, roll a dice



    - Discrete-time, continuous-amplitude
      - Example: samples of analog signal, average monthly temperature



SIGNAL CLASSIFICATION

- **Periodic signal v.s. aperiodic signal**
  - Periodic signal $x(n) = x(n+N)$
    - The smallest value of *N* that satisfies this relation is the fundamental periods.
  - Is $\cos(\omega n)$ periodic?

$\cos(\omega n)$ is periodic if $\frac{2k\pi}{\omega} = N$ is an integer, and the smallest N is the fundamental period.

- Example: $\cos(3n)$

  $\cos(\pi n)$

  $\cos(\frac{3}{4}\pi n)$

SIGNAL CLASSIFICATION

- **Sum of two periodic signals**

$x_1(n)$: fundamental period N_1

$x_2(n)$: fundamental period N_2

$$x_1(n)+x_2(n)$$

$x_1(n)+x_2(n)$ is periodic if both $x_1(n)$ and $x_2(n)$ are periodic.
Assume

$$\frac{N_1}{N_2}=\frac{p}{q}$$

where p and q are not divisible of each other. The period is $N=pN_2=qN_1$

SIGNAL CLASSIFICATION

- **Example:**
  - Is the signal periodic? If it is, what is the fundamental period?

$$\cos(\frac{\pi n}{9}) + \sin(\frac{3\pi n}{7} + \frac{1}{2})$$

SIGNAL CLASSIFICATION

- **Energy signal**
  - Energy:

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

  - Review: energy of continuous-time signal

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

  - Energy signal: $0 < E < \infty$

SIGNAL CLASSIFICATION

- **Power signal**
  - Power of discrete-time signal

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

  - Review: power of continuous-time signal

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

  - Power signal: $0 < P < \infty$

SIGNAL CLASSIFICATION

- **Example**
  - Determine if the discrete-time exponential signal is an energy signal or power signal

$$x(n) = 2(0.5)^n \qquad n \geq 0$$

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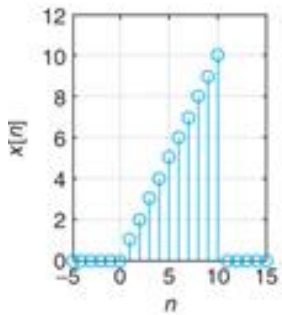
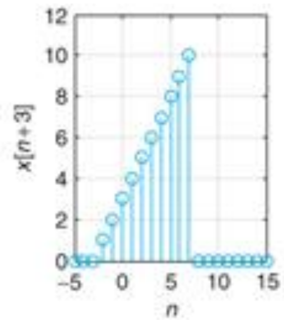
ELEMENTARY SIGNALS

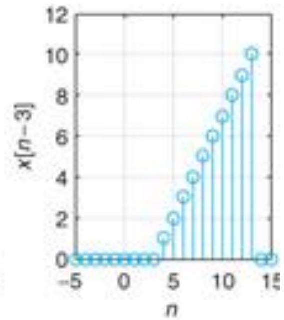
- **Basic signal operations**
  - Time shifting $x(n-k)$
    - shift the signal to the right by *k* samples



$x(n)$

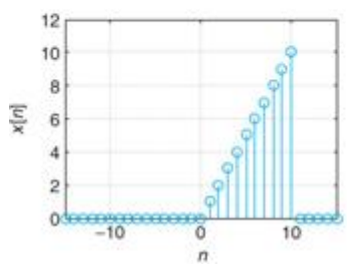


$x(n+3)$

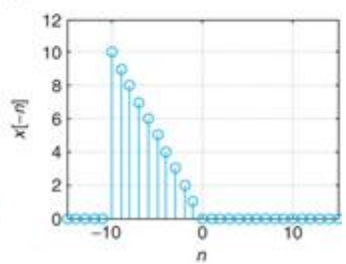


$x(n-3)$

  - Reflecting $x(-n)$
    - Reflecting $x(n)$ with respect to n = 0.



$x(n)$



$x(-n)$

ELEMENTARY SIGNALS

- **Basic signal operations**
  - Time scaling
    - Example: Let $x(n) = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$

      Find $x(2n)$, $x(2n+1)$

ELEMENTARY SIGNALS

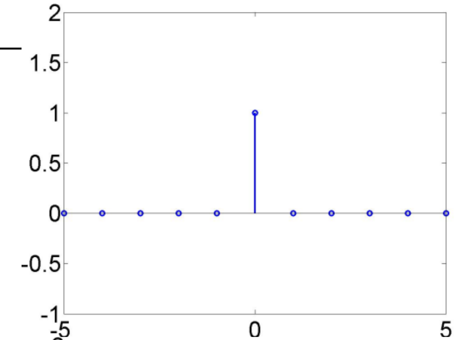
- **Basic signal operations**
  - Time scaling
    - Example: $x(n) = [-1,\underset{\uparrow}{2},1,0,-2]$

the $\uparrow$ always points to $x(0)$

find $x(3n), x\left(\frac{n}{3}\right), x\left(\frac{n}{3}+\frac{2}{3}\right)$

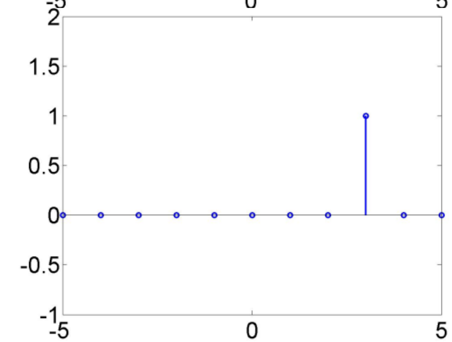
ELEMENTARY SIGNALS

- **Unit impulse function**



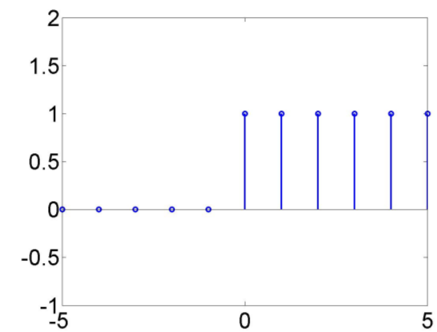
$$\delta(n)=\begin{cases}1, n=0,\\ 0, n\neq 0.\end{cases}$$

  – time shifting



$$\delta(n-k)=\begin{cases}1, n=k,\\ 0, n\neq k.\end{cases}$$

- **Unit step function**



$$u(n)=\begin{cases}0, n<0,\\ 1, n\geq 0.\end{cases}$$

- **Relation between unit impulse function and unit step function**

$$\delta(n)=u(n)-u(n-1)$$

$$u(n)=\sum_{k=-\infty}^{n}\delta(k)$$

ELEMENTARY SIGNALS

- **Exponential function**

$$x(n) = \exp(\alpha n)$$

- **Complex exponential function**

$$x(n) = \exp(j\omega_0 n) = \cos(\omega_0 n) + j\sin(\omega_0 n)$$

  – $x(n)$ is periodic if $\frac{2k\pi}{\omega_0} = N$ is an integer, and the smallest integer N is the fundamental period.

- **Example**

  – Are the following signals periodic? If periodic, find fundamental period.

$$x_1(n) = \exp\left(j\frac{7\pi}{9}n\right) \qquad x_2(n) = \exp\left(j\frac{7}{9}n\right)$$

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- **Linear time-invariant (LTI) discrete-time systems**
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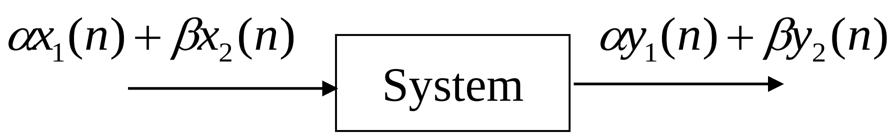
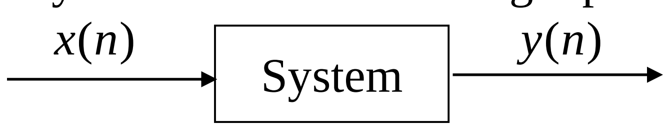
DISCRETE-TIME SYSTEMS

- **Linear system**
  - Consider a system with the following input-output relationship

  - The system is linear if it meets the superposition principle



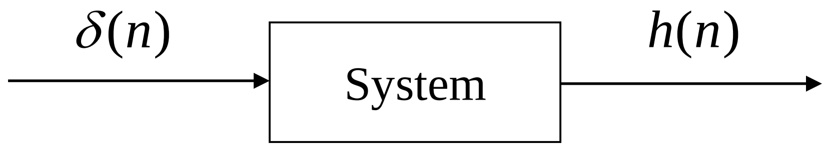
- **Time-invariant system**
  - Consider a system with the following input-output relationship



  - The system is time-invariant if a time-shift at the input leads to the same time-shift at the output

DISCRETE-TIME SYSTEM

- **Impulse response of a LTI system**
  - The response of the system when the input is $\delta(n)$



- **Any arbitrary discrete-time signal can be decomposed as weighted summation of the unit impulse functions**

$$x(n) = \sum_{k=-\infty}^{+\infty} x(k)\delta(n-k)$$

  - Why?

    E.g. $x(3) = \sum_{k=-\infty}^{+\infty} x(k)\delta(3-k) =$

  - Recall: $x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$

DISCRETE-TIME SYSTEM

- **LTI response to arbitrary input**
  - Any arbitrary signal can be written as

$$x(n) = \sum_{k=-\infty}^{+\infty} x(k)\delta(n-k)$$

  - Time-invariant

$$\delta(n-k) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(n-k)$$

  - Linear

$$\sum_{k=-\infty}^{+\infty} x(k)\delta(n-k) \longrightarrow \boxed{\text{LTI}} \longrightarrow \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

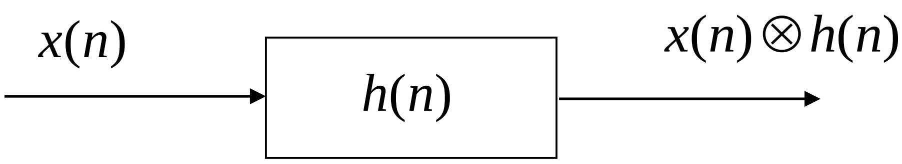
$$x(n) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

DISCRETE-TIME SYSTEM

- **Convolution sum**
  - The convolution sum of two signals $x(n)$ and $h(n)$ is

$$x(n) \otimes h(n) = \sum_{k=-\infty}^{+\infty} x(k) h(n-k)$$

- **Response of LTI system**
  - The output of a LTI system is the convolution sum of the input and the impulse response of the system.



DISCRETE-TIME SYSTEM

- **Examples**
  - 1. $x(n) \otimes \delta(n-m)$
  - 2. $x(n) = \alpha^n u(n), \qquad h(n) = \beta^n u(n)$

    $x(n) \otimes h(n) =$

DISCRETE-TIME SYSTEM

- **Example**
  - 3. Find the step response of the system with impulse response

$$h(n) = 2\left(\frac{1}{2}\right)^{n} \cos\left(\frac{2\pi}{3} n\right) u(n)$$

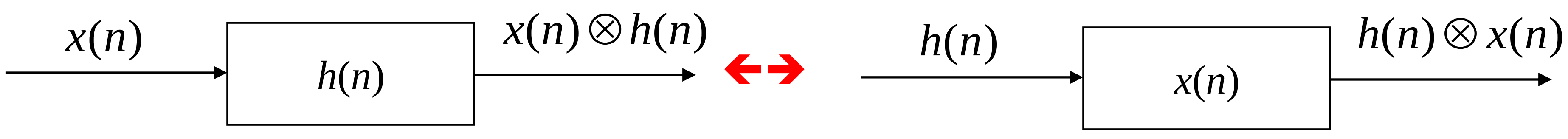
DISCRETE-TIME SYSTEM

- **Example:**
  - Let $x(n) = [\underset{\uparrow}{0},1,2]$ $h(n) = [-1,\underset{\uparrow}{-2},-3,-4]$, be two sequences, find $x(n) \otimes h(n)$

DISCRETE-TIME SYSTEM

- **Properties: commutativity**

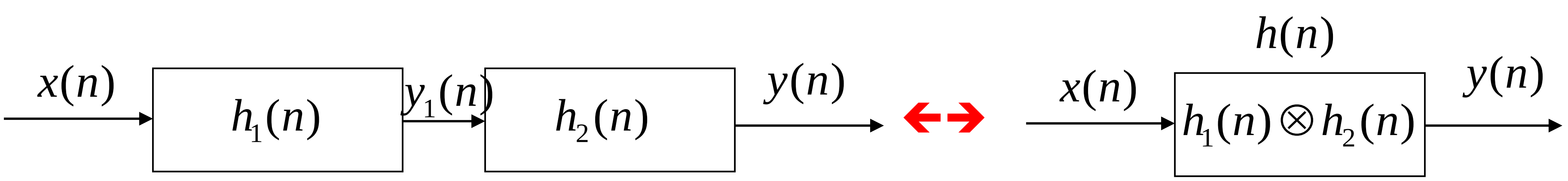
$$x(n) \otimes h(n) = h(n) \otimes x(n)$$



DISCRETE-TIME SYSTEM

- **Properties: associativity**

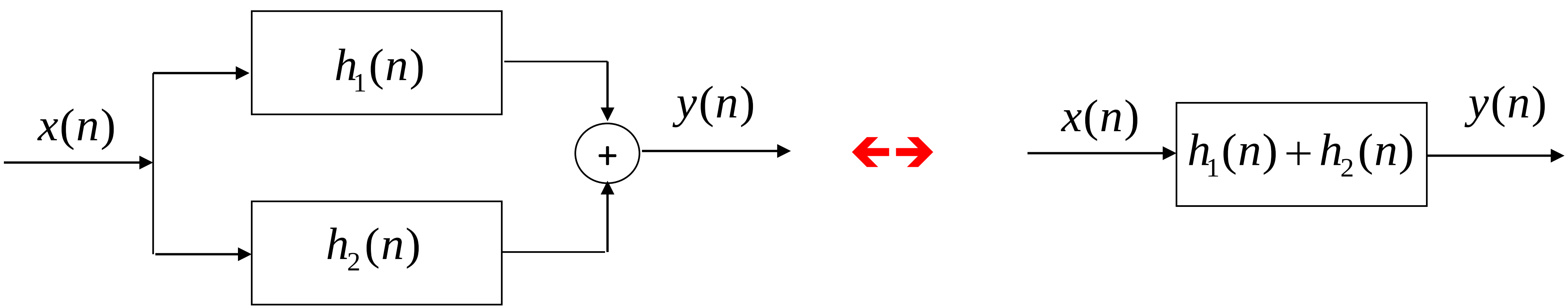
$$x(n)\otimes h_1(n)\otimes h_2(n)=\left[x(n)\otimes h_1(n)\right]\otimes h_2(n)=x(n)\otimes\left[h_1(n)\otimes h_2(n)\right]$$



DISCRETE-TIME SYSTEM

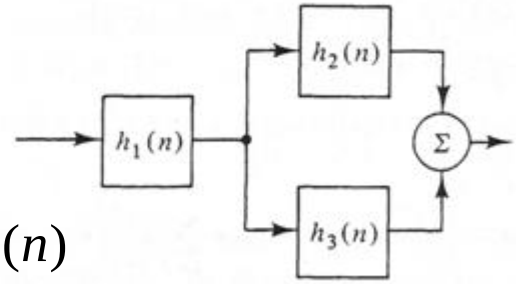
- **Distributivity**

$$x(n) \otimes \left[h_1(n) + h_2(n)\right] = \left[x(n) \otimes h_1(n)\right] + \left[x(n) \otimes h_1(n)\right]$$



DISCRETE-TIME SYSTSEMS

- **Example**
  - Consider a system shown in the figure. Find the overall impulse response.



$h_1(n) = \delta(n) - 2\delta(n-1)$ $\quad h_2(n) = (n-1)u(n)$ $\quad h_3(n) = 2^n u(n)$

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- **Causality and stability**
- Difference equation representation of LTI systems

CAUSALITY AND STABILITY

- **Causal system**
  - A discrete-time system is causal if the output $y(n_0)$ depends only on values of input for $n \le n_0$
    - The output does not depend on future input.
  - Example:
    - determine whether the following systems are causal.

$$y(n) = x^2(n) + 3x(n)$$

$$y(n) = x^2(n-1)$$

$$y(n) = \frac{1}{3}\left[x(n-1) + x(n) + x(n+1)\right]$$

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

$$y(n) = \sum_{k=-\infty}^{n+1} x(k)$$

CAUSALITY AND STABILITY

- **Causality of LTI system**
  - An LTI system is causal if and only if $h(n)=0$ for $n < 0$
    - Why?

$$y(n)=\sum_{-\infty}^{+\infty}x(k)h(n-k)=\sum_{-\infty}^{n}x(k)h(n-k)+\sum_{n+1}^{+\infty}x(k)h(n-k)$$

  - A signal $x(n)$ is causal if $x(n)=0$ for $n < 0$.
  - Example:
    - For LTI systems with impulse responses given as follows. Find if the systems are casual.

$$h(n)=\cos(2n)$$

$$h(n)=\cos(2n)u(n)$$

$$h(n)=a^{n}u(n)+b^{n}u(n+1)$$

$$h(n)=a^{n}u(n)+b^{n}u(n-1)$$

CAUSALITY AND STABILITY

- **Bounded-input bounded-output (BIBO) stable**
  - a system is BIBO stable if, for any bounded input $x(n)$, the response $y(n)$ is also bounded.
- **BIBO stability of LTI system**
  - An LTI discrete-time system is BIBO stable if

$$\sum_{-\infty}^{+\infty} |h(k)| < \infty$$

    - Why?

CAUSALITY AND STABILITY

- **Example**
  - For an LTI system with impulse responses as follows. Are they BIBO stable?

$$h(n) = (0.5)^n u(n)$$

$$h(n) = (-0.5)^n u(n)$$

$$h(n) = (0.5)^n$$

$$h(n) = 2^n u(n)$$

$$h(n) = 2^n u(-n)$$

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DIFFERENCE EQUATION

- **Difference equation representation of LTI discrete-time system**
  - Any LTI discrete-time system can be represented as

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

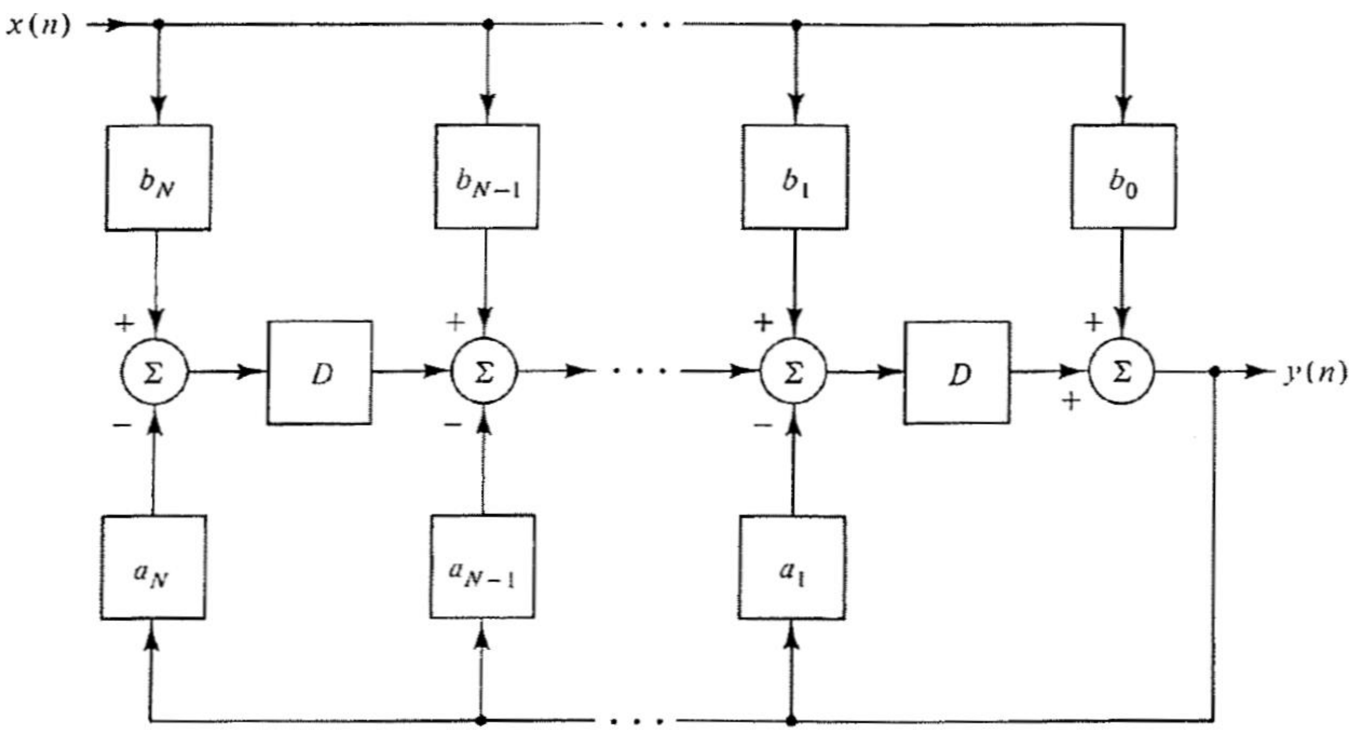
  - Review: any LTI continuous-time system can be represented as

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

DIFFERENCE EQUATION

- **Simulation diagram**

$$y(n)+a_1 y(n-1)+\cdots+a_N y(n-N)=b_0 x(n)+b_1 x(n-1)+\cdots+b_N x(n-N)$$



DIFFERENCE EQUATION

- **Example**
  - Draw the simulation diagram of the LTI system described by the following difference equation

$$2y(n)+3y(n-1)=0.5x(n)+x(n-1)+5x(n-2)$$

DIFFERENCE EQUATION

- **Example**
  - The impulse response of an LTI system is $h(n) = [\underset{\uparrow}{2},3,0,5]$
    - Find the difference equation representation
    - Draw the simulation diagram.