Department of Electrical Engineering
University of Arkansas

ELEG 3143 Probability & Stochastic Process
Ch. 6 Stochastic Process

Dr. Jingxian Wu

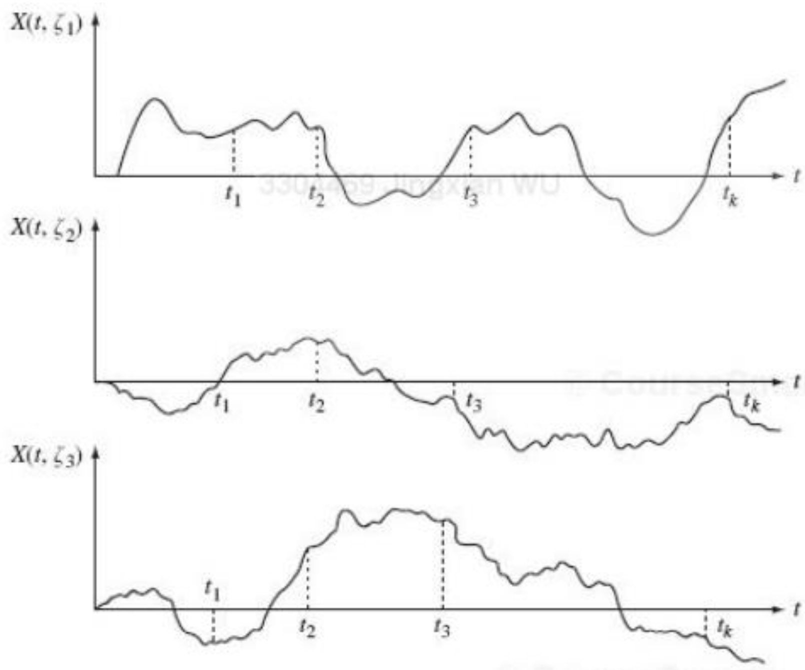
wuj@uark.edu

OUTLINE

- Definition of stochastic process (random process)
- Description of continuous-time random process
- Description of discrete-time random process
- Stationary and non-stationary
- Two random processes
- Ergodic and non-ergodic random process
- Power spectral density

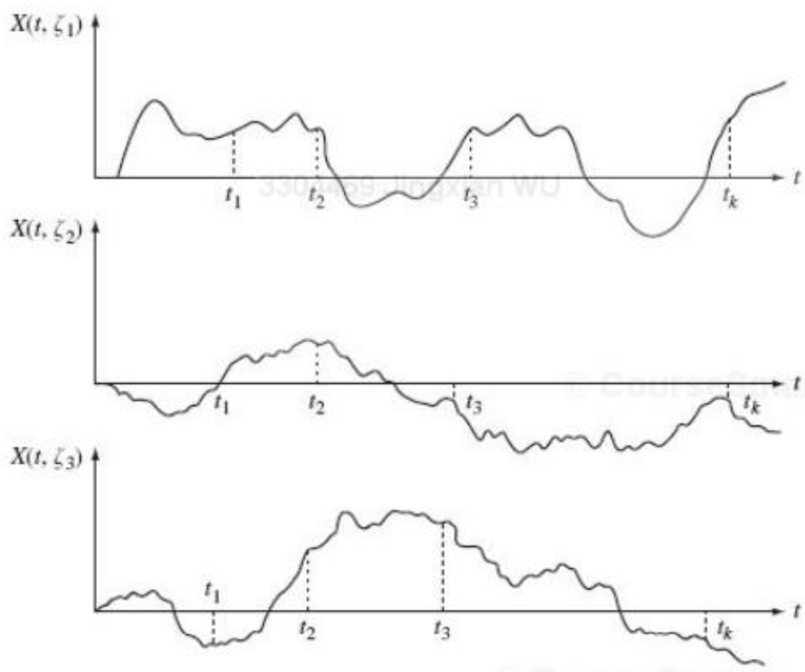
DEFINITION

- **Stochastic process, or, random process**
  - A random variable changes with respect to time
    - Example: the temperature in the room
      - At any given moment, the temperature is random: random variable
      - The temperature (the value of the random variable) changes with respect to time.



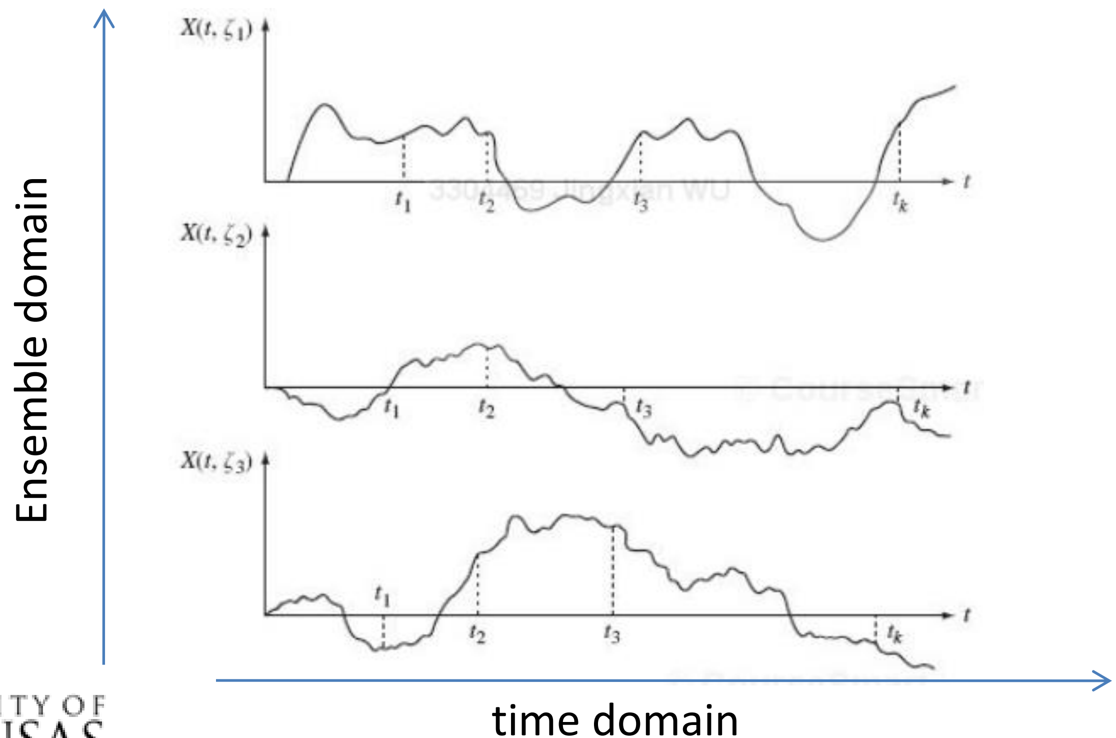
DEFINITION

- **Stochastic process (random process)**
  - Recall: Random variable is a mapping from random events to real number: $X(\xi)$, where ξ is a random event
  - Stochastic process is a random variable changes with time
    - Denoted as: $X(t,\xi)$
    - It is a function of random event ξ , and time *t*.
    - Recall: random variable is a function of random event ξ



DEFINITION

- **Stochastic process (random process)** $X(t,\xi)$
  - Fix time: $X(t_k,\xi)$ is a random variable
    - pdf, CDF, mean, variance, moments, etc.
  - fix event: $X(t,\xi_k)$ is a deterministic time function
    - Defined as a realization of a random process
    - Or: a sample function of a random process



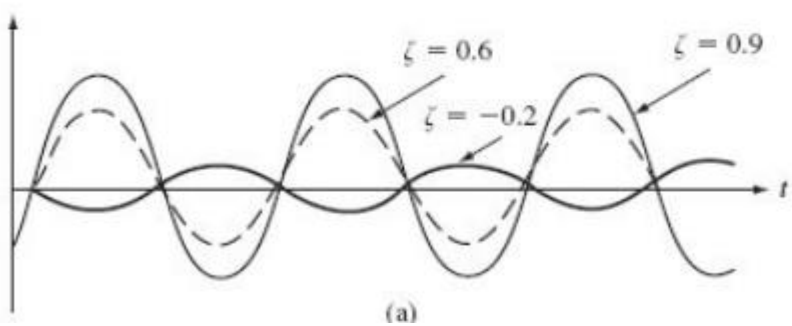
DEFINITION

- **Example**
  - Let ξ be a Gaussian random variable with 0 mean and unit variance, then

$$X(t,\xi) = \xi \cdot \cos(2\pi f t)$$

is a random process .

  - If we fix $t = t_0$, we get a Gaussian random variable $X(t_0,\xi) = \xi \cdot \cos(2\pi f t_0)$
    - The pdf of $X(t_0,\xi)$ is:
  - If we fix $\xi = \xi_0$, we get sample function $X(t,\xi_0) = \xi_0 \cdot \cos(2\pi f t)$
    - It is a deterministic function of time



(a)

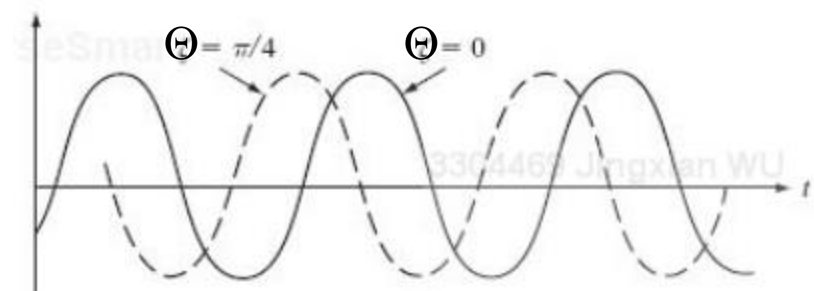
UNIVERSITY OF ARKANSAS

DEFINITION

- **Stochastic process:**
  - A random variable changes with respect to time $X(t,\xi)$
- **Sample function:**
  - A deterministic time function $X(t,\xi_k)$ associated with outcome ξ_k
- **Ensemble:**
  - The set of all possible time functions of a stochastic process.

DEFINITION

- **Example**
  - Let Θ be a random variable uniformly distributed between $[-\pi, \pi]$, then $X(t, \Theta) = A\cos(2\pi f t + \Theta)$ is a random process, where A is a constant. At $t = t_0$ Find the mean and variance of $X(t_0, \Theta)$



(b)

DEFINITION

- **Example**
  - Let Θ be a random variable uniformly distributed between $[-\pi,\pi]$, and ξ a Gaussian random variable with 0 mean and variance σ^2. Then $\xi\cos(2\pi ft+\Theta)$ is a random process. ξ and Θ are independent. Find the mean and variance of the random process at $t=t_0$

DEFINITION

- **Continuous-time random process**
  - The time is continuous
  - E.g. $X(t,\xi) = \xi \cdot \cos(2\pi f t)$ $\quad t \in R$
- **Discrete-time random process**
  - The time is discrete
  - E.g. $X(n,\xi) = \xi \cdot \cos(2\pi f n)$ $\quad n = 0,1,2,\Lambda$
- **Continuous random process**
  - At any time, $X(t_0,\xi)$ is a continuous random variable
  - E.g. $X(t,\xi) = \xi \cdot \cos(2\pi f t)$, ξ is Gaussian distributed
- **Discrete random process**
  - At any time, $X(t_0,\xi)$ is a discrete random variable
  - E.g. $X(t,\xi) = \xi \cdot \cos(2\pi f t)$, ξ is a Bernoulli RV

DEFINITION

- **Classifications**
  - Continuous-time v.s. discrete-time
  - Continuous v.s. discrete
  - Stationary v.s. non-stationary
  - Wide sense stationary (WSS) v.s. non-WSS
  - Ergodic v.s. non-ergodic

- **Simple notation**
  - Random variable: $X(\xi)$, usually denoted as X
  - Random process: $X(t,\xi)$, usually denoted as $X(t)$

OUTLINE

- Definition of stochastic process (random process)
- **Description of continuous-time random process**
- Description of discrete-time random process
- Stationary and non-stationary
- Two random processes
- Ergodic and non-ergodic random process
- Power spectral density

DESCRIPTION

- **How do we describe a random process?**
  - A random variable is fully characterized by its pdf or CDF.
  - How do we statistically describe random process?
    - E.g.

$$\Pr(a_1 < X(t_1) \leq b_1, a_2 < X(t_2) \leq b_2) = ?$$

$$\Pr(a < X(t_{k+1}) \leq b | X(t_1) = a_1, \cdots, X(t_k) = a_k) = ?$$

  - We need to describe the random process in both the ensemble domain and the time domain

- **Descriptions of the random process**
  - Joint distribution of time samples: joint pdf, joint CDF
  - Moments: mean, variance, correlation function, covariance function

DESCRIPTION

- **Joint distribution of time samples**
  - Consider a random process $X(t)$
  - Let $X_1 = X(t_1), X_2 = X(t_2), \cdots, X_n = X(t_n)$
  - joint CDF:

$$F_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = \Pr(X_1 \leq x_1,\cdots,X_n \leq x_n)$$

  - Joint pdf:

$$f_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = \frac{d^n(X_1 \leq x_1,\cdots,X_n \leq x_n)}{dx_1\cdots dx_n}$$

  - A random process is fully specified by the collections of all the joint CDFs (or joint pdfs) for any n and any choice of sampling instants.
    - For a continuous-time random process, there will be infinite such joint CDFs.

DESCRIPTION

- **Moments of time samples**
  - Provide a partial description of the random process
  - For most practical applications it is sufficient to have a partial description.
- **Mean function**

$$m_X(t) = E[X(t)] = \int_{-\infty}^{+\infty} u f_{X(t)}(u)\,du$$

  - The mean function is a deterministic function of time.
- **Variance function**

$$\sigma_X^2(t) = E\left\{[X(t) - m_X(t)]^2\right\} = \int_{-\infty}^{+\infty} [x - m_X(t)]^2 f_{X(t)}(x)\,dx$$

UNIVERSITY OF ARKANSAS

DESCRIPTION

- **Example**
  - For a random process $X(t) = A \cdot Cos(2\pi ft)$, where A is a random variable with mean m_A and variance σ_A^2. Find the mean function and variance function of $X(t)$.

DESCRIPTION

- **Autocorrelation function (ACF)**
  - The autocorrelation function of a random process X(t) is defined as the

$$R_X(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$

  - It describes the correlation of the random process in the time domain
    - How are two events happened at different times related to each other.
      - E.g. if there is a strong correlation between the temperature today and the temperature tomorrow, then we can predict tomorrow's temperature by using today's observation.
      - E.g. the text in a book can be considered as a discrete random process
        - » Stochasxxx
        - » We can easily guess the contents in xxx by using the time correlation.
  - $R_X(t_1,t_1) = E\left[X^2(t_1)\right]$ is the second moment of $X(t_1)$

DESCRIPTION

- **Autocovariance function**
  - The autocovariance function of a random process X(t) is defined as the

$$C_X(t_1,t_2) = E\{[X(t_1) - m_X(t_1)][X(t_2) - m_X(t_2)]\}$$

$$= E[X(t_1)X(t_2)] - m_X(t_1)m_X(t_2)$$

  - $C_X(t,t) = \sigma^2_{X(t)}$

DESCRIPTION

- **Example**
  - Consider a random process $X(t) = A \cdot \cos(2\pi ft)$, where A is a random variable with mean m_A and variance σ_A^2. Find the autocorrelation function and autocovariance function.

DESCRIPTION

- **Example**
  - Consider a random process $X(t) = A \cdot \cos(2\pi f t + \Theta)$ where A is a random variable with mean m_A and variance σ_A^2 . Θ is uniformly distributed in $[-\pi, \pi]$. A and Θ are independent.
  - Find the autocorrelation function and autocovariance function.

DESCRIPTION

- **Example**
  - Given a random process $X(t)$ with expected value $\mu_X(t)$ and autocorrelation $R_X(t,\tau)$, we can make the noisy observation $Y(t) = X(t) + N(t)$ where $N(t)$ is a random noise process with $\mu_N(t) = 0$ and autocorrelation $R_N(t,\tau)$. Assuming that the noise process $N(t)$ is independent of $X(t)$, nd the expected value and autocorrelation of $Y(t)$.

OUTLINE

- Definition of stochastic process (random process)
- Description of continuous-time random process
- **Description of discrete-time random process**
- Stationary and non-stationary
- Two random processes
- Ergodic and non-ergodic random process
- Power spectral density

DESCRIPTION

- **Discrete-time random process (random sequence)**
  - A discrete-time random process

$$\cdots, X_{-1}, X_0, X_1, X_2, \cdots, X_n, \cdots$$

  - Consider a subset of m samples $\quad \mathbf{X} = [X_{n_1}, X_{n_2}, \cdots, X_{n_m}]$
    - E.g. $\quad \mathbf{X} = [X_2, X_5, X_8, X_{10}]$
  - Joint PMF

$$p_{\mathbf{X}}(x_1, \cdots, x_m) = \Pr(X_{n_1} = x_1, \cdots, X_{n_m} = x_m)$$

  - Joint CDF

$$F_{\mathbf{X}}(x_1, \cdots, x_m) = \Pr(X_{n_1} \leq x_1, \cdots, X_{n_m} \leq x_m)$$

DESCRIPTION

- **Bernoulli process**
  - A Bernoulli (p) process X_n is an indepdnent and identically distributed (i.i.d.) random sequence in which each X_n is a Bernoulli (p) random variable.

    1. Find the joint PMF of $\mathbf{X} = [X_1, \cdots, X_n]$.
    2. Find the joint PMF of $\mathbf{X} = [1, 0, 0, 1, 1, 0, 0, 0, 1]$ with $p = 0.3$.

DESCRIPTION

- **Moments of time samples**
  - Provide a partial description of the random process
  - For most practical applications it is sufficient to have a partial description.
- **Mean function**

$$m_X(n) = \mathbb{E}[X_n]$$

  - The mean function is a deterministic function of time.
- **Variance function**

$$\sigma_X^2(n) = \mathbb{E}[(X_n - m_X(n))^2] = \mathbb{E}[X_n^2] - m_X^2(n)$$

  - It is a deterministic function of time

DESCRIPTION

- **Example**
  - For a Bernoulli (p) process, find the mean function and variance function.

DESCRIPTION

- **Autocorrelation function of a random sequence**

$$R_X[m,k] = E\left[X_m X_{m+k}\right].$$

- **Autocovariance function of a random sequence**

$$C_X[m,k] = \mathrm{Cov}\left[X_m, X_{m+k}\right]$$

$$C_X[n,k] = R_X[n,k] - \mu_X(n)\mu_X(n+k).$$

DESCRIPTION

- **Example**

The input to a digital filter is an iid random sequence $\ldots, X_{-1}, X_0, X_1, \ldots$ with $E[X_i] = 0$ and $\mathrm{Var}[X_i] = 1$. The output is a random sequence $\ldots, Y_{-1}, Y_0, Y_1, \ldots$, related to the input sequence by the formula

$$Y_n = X_n + X_{n-1} \qquad \text{for all integers } n. \tag{10.38}$$

Find the expected value $E[Y_n]$ and autocovariance function $C_Y[m, k]$.

OUTLINE

- Definition of stochastic process (random process)
- Description of continuous-time random process
- Description of discrete-time random process
- **Stationary and non-stationary**
- Two random processes
- Ergodic and non-ergodic random process
- Power spectral density

STATIONARY RANDOM PROCESS

- **Stationary random process**
  - A random process, X(t), is stationary if the joint distribution of any set of samples does not depend on the time origin.

$$f_{X(t_1),\ldots,X(t_m)}(x_1,\ldots,x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1,\ldots,x_m)$$

For any value of τ and m, and for any choice of $t_1, t_2, \cdots, t_m$

STATIONARY RANDOM PROCESS

- **1^{st} order distribution**
  - If X(t) is a stationary random process, then the first order CDF or pdf must be independent of time

$$F_{X(t)}(x) = F_{X(t+\tau)}(x), \qquad \forall t, \tau$$

$$f_{X(t)}(x) = f_{X(t+\tau)}(x), \qquad \forall t, \tau$$

    - The samples at different time instant have the same distribution.
  - Mean:

$$E[X(t)] =$$

$$E[X(t+\tau)] =$$

For a stationary random process, the mean is independent of time

STATIONARY RANDOM PROCESS

- **2^{nd} order distribution**
  - If X(t) is a stationary random process, then the 2^{nd} order CDF and pdf are:

$$F_{X(t)X(t+\tau)}(x_1, x_2) = F_{X(0)X(\tau)}(x_1, x_2), \qquad \forall t, \tau$$

$$f_{X(t)X(t+\tau)}(x_1, x_2) = f_{X(0)X(\tau)}(x_1, x_2), \qquad \forall t, \tau$$

  - The 2^{nd} order distribution only depends on the time difference between the two samples
  - Autocorrelation function

$$R_X(t_1, t_2) =$$

  - Autocovariance function:

For a stationary random process, the autocorrelation function and autocovariance function only depends on the time difference: $t_2 - t_1$

STATIONARY RANDOM PROCESS

- **Stationary random process**
  - In order to determine whether a random process is stationary, we need to find out the joint distribution of any group of samples
  - Stationary random process → the joint distribution of any group of time samples is independent of the starting time
  - This is a very strict requirement, and sometimes it is difficult to determine whether a random process is stationary.

STATIONARY RANDOM PROCESS

- **Wide-Sense Stationary (WSS) Random Process**
  - A random process, X(t), is wide-sense stationary (WSS), if the following two conditions are satisfied
    - The first moment is independent of time

$$E\left[X(t)\right] = E[X(t+\tau)] = m_X$$

    - The autocorrelation function depends only on the time difference

$$E\left[X(t)X(t+\tau)\right] = R_X(\tau)$$

  - Only consider the 1st order and 2nd order distributions

STATIONARY RANDOM PROCESS

- **Stationary v.s. Wide-Sense Stationary**
  - If X(t) is stationary ➔ X(t) is WSS
    - It is not true the other way around
  - Stationary is a much stricter condition. It requires the joint distribution of any combination of samples to be independent of the absolute starting time

$$f_{X(t_1),\ldots,X(t_m)}(x_1,\ldots,x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1,\ldots,x_m)$$

  - WSS only considers the first moment (mean is a constant) and second order moment (autocorrelation function depends only on the time difference)

$$E\left[X(t)\right] = E[X(t+\tau)] = m_X$$

$$E\left[X(t)X(t+\tau)\right] = R_X(\tau)$$

STATIONARY RANDOM PROCESS

- **Example**
  - A random process is described by

$$X(t) = A + B\cos(2\pi ft + \Theta)$$

where A is a random variable uniformly distributed between [-3, 3], B is an RV with zero mean and variance 4, and Θ is a random variable uniformly distributed in $[-\pi/2, 3\pi/2]$. A, B, and Θ are independent. Find the mean and autocorrelation function. Is X(t) WSS?

STATIONARY RANDOM PROCESS

- **Autocorrelation function of a WSS random process**

$$R_X(\tau) = E\left[X(t)X(t+\tau)\right]$$

– $R_X(0) = E\left[X^2(t)\right]$ is the average power of the signal X(t)

– $R_X(\tau) = E\left[X(t)X(t+\tau)\right] = E\left[X(t+\tau)X(t)\right] = R_X(-\tau)$ is an even function

– $\left|R_X(\tau)\right| \le R_X(0)$

- Cauchy-Schwartz inequality $E[XY]^2 \le E[X^2]E[Y^2]$,

– If $R_X(\tau)$ is non-periodic, then

$$\lim_{\tau\to\infty} R_X(\tau) = m_X^2$$

STATIONARY RANDOM PROCESS

- **Example**
  - Consider a random process having an autocorrelation function

$$R_X(\tau) = 3\frac{\tau^2 + 4}{\tau^2 + 3}$$

    - Find the mean and variance of X(t)

STATIONARY RANDOM PROCESS

- **Example**
  - A random process Z(t) is

$$Z(t) = X(t) + X(t + t_0)$$

Where X(t) is a WSS random process with autocorrelation function

$$R_Z(\tau) = \exp(-\tau^2)$$

Find the autocorrelation function of Z(t). Is Z(t) WSS?

STATIONARY RANDOM PROCESS

- **Example**
  - A random process $X(t) = At + B$, where A is a Gaussian random variable with 0 mean and variance 16, and B is uniformly distributed between 0 and 6. A and B are independent. Find the mean and auto-correlation function. Is it WSS?

OUTLINE

- Definition of stochastic process (random process)
- Description of continuous-time random process
- Description of discrete-time random process
- Stationary and non-stationary
- **Two random processes**
- Ergodic and non-ergodic random process
- Power spectral density

TWO RANDOM PROCESSES

- **Two random processes**
  - X(t) and Y(t)
- **Cross-correlation function**
  - The cross-correlation function between two random processes X(t) and Y(t) is defined as

$$R_{XY}(t_1,t_2)=E[X(t_1)Y(t_2)]$$

  - Two random processes are said to be uncorrelated if for all t_1 and t_2

$$E[X(t_1)Y(t_2)]=E[X(t_1)]E[Y(t_2)]$$

  - Two random processes are said to be orthogonal if for all t_1 and t_2

$$R_{XY}(t_1,t_2)=0$$

- **Cross-covariance function**
  - The cross-covariance function between two random processes X(t) and Y(t) is defined as

$$C_{XY}(t_1,t_2)=E[X(t_1)Y(t_2)]-m_X(t_1)m_X(t_2)$$

TWO RANDOM PROCESSES

- **Example**
  - Consider two random processes $X(t) = \cos(2\pi ft + \Theta)$ and $Y(t) = \sin(2\pi ft + \Theta)$. Are they uncorrelated?

TWO RANDOM PROCESSES

- **Example**
  - Suppose signal Y(t) consists a desired signal X(t) plus noise N(t) as

    Y(t) = X(t) + N(t)

    The autocorrelation functions of X(t) and N(t) are: $R_{XX}(t_1,t_2)$ and $R_{NN}(t_1,t_2)$, respectively. The mean function of X(t) and N(t) are: $m_X(t)$ and $m_N(t)$, respectively. Find the cross-correlation between X(t) and Y(t).

OUTLINE

- Definition of stochastic process (random process)
- Description of continuous-time random process
- Description of discrete-time random process
- Stationary and non-stationary
- Two random processes
- **Ergodic and non-ergodic random process**
- Power spectral density

ERGODIC RANDOM PROCESS

- **Time average of a signal *x*(*t*)**

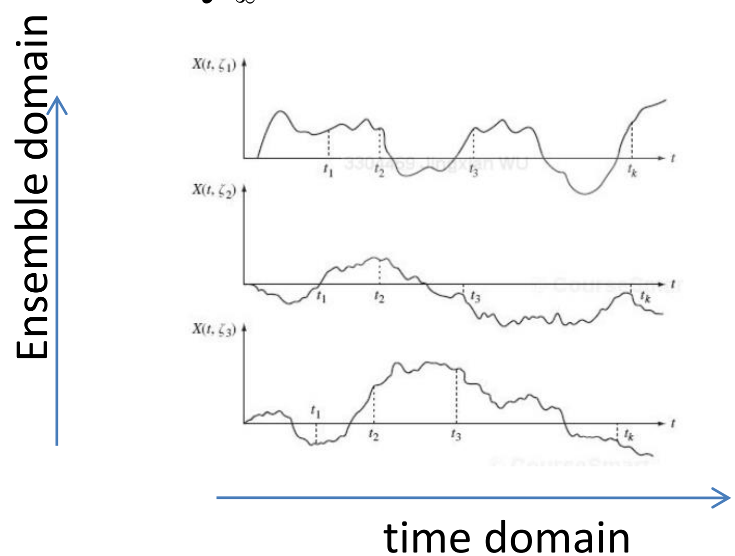
$$\langle x(t)\rangle_T = \frac{1}{T}\int_0^T x(t)dt$$

- **Time average of a sample function of the random process** $x(t) = X(t, \xi_0)$

$$\langle X(t,\xi_0)\rangle_T = \frac{1}{T}\int_0^T X(t,\xi_0)dt$$

- **Ensemble average (mean) of a random process** $X(t, \xi_0)$

$$E\left[X(t,\xi)\right] = \int_{-\infty}^{+\infty} x f_{X(t,\xi)}(x)dx$$



ERGODIC RANDOM PROCESS

- **Ergodic random process**
  - A stationary random process is also an ergodic random process if the n-th order ensemble average is the same as the n-th order time average.

$$\boxed{\left\langle X^n(t) \right\rangle = E\left[X^n(t)\right]}$$

$$\left\langle x^n(t) \right\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^n(t)dt$$

$$E\left[X^n(t)\right] = \int_{-\infty}^{+\infty} x^n f_{X(t)}(x)dx$$

  - If a random process is ergodic, we can find the moments by performing time average over a single sample function.
  - Ergodic is only defined for stationary random process
    - If a random process is ergodic , then it must be stationary
      - Not true the other way around

ERGODIC RANDOM PROCESS

- **Example**
  - Consider a random process $X(t) = A$

    where A is a Gaussian RV with 0 mean and variance 4. Is it ergodic?

ERGODIC RANDOM PROCESS

- **Mean ergodic**
  - A WSS random process is mean ergodic if the ensemble average is the same as the time average.

$$\langle X(t) \rangle = E[X(t)]$$

  - Mean ergodic v.s. ergodic
    - Ergodic: $\langle X^n(t) \rangle = E[X^n(t)]$ for stationary process
    - Mean ergodic: $\langle X(t) \rangle = E[X(t)]$ for WSS process
    - If a random process is ergodic, then it must be mean ergodic
      - Not true the other way around
  - If a process is mean ergodic, it must be WSS
    - Mean ergodic is defined for WSS process only.

UNIVERSITY OF ARKANSAS

ERGODIC RANDOM PROCESS

- **Example**
  - Consider a random process $X(t) = A\cos(2\pi ft + \Theta)$ where A is a non-zero constant, and Θ is uniformly distributed between 0 and π. Is it mean ergodic?

OUTLINE

- Definition of stochastic process (random process)
- Description of continuous-time random process
- Description of discrete-time random process
- Stationary and non-stationary
- Two random processes
- Ergodic and non-ergodic random process
- **Power spectral density**

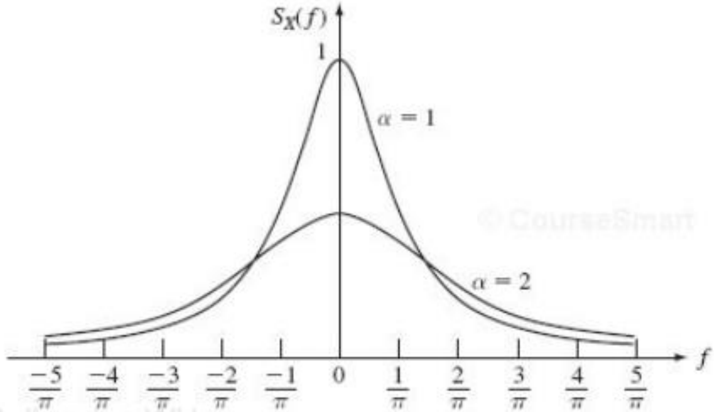
POWER SPECTRAL DENSITY

- **Power spectral density (PSD)**
  - The distribution of the power in the frequency domain.
  - For a WSS random process, the PSD is the Fourier transform of the auto-correlation function

$$S_X(f) = F[R_X(\tau)]$$

  - The "density" of power in the frequency domain.
  - The power consumption between the frequency range $[f_1, f_2]$:

$$P = \int_{f_1}^{f_2} S_X(f)df$$



POWER SPECTRAL DENSITY

- **White noise**
  - Autocorrelation function

$$R_x(\tau) = \frac{N_0}{2}\delta(\tau)$$

    - any two samples in the time domain are uncorrelated.
  - Power spectral density

$$S_x(f) = \int_{-\infty}^{+\infty} \frac{N_0}{2}\delta(\tau)d\tau = \frac{N_0}{2}$$