Department of Electrical Engineering
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ELEG 3143 Probability & Stochastic Process
Ch. 4 Multiple Random Variables

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OUTLINE

- **Two discrete random variables**
- **Two continuous random variables**
- **Statistical independence and correlation**
- **Functions of two random variables**
- **Moment generating function**

TWO DISCRETE RANDOM VARIABLES

- **Joint PMF of two discrete random variables**
  - Consider two discrete RVs, X and Y. The joint probability mass function (PMF) of X and Y is defined as

$$p_{XY}(x_i, y_j) = \Pr(X = x_i, Y = y_j) = \Pr\left(\{X = x_i\} \cap \{Y = y_j\}\right)$$

TWO DISCRETE RANDOM VARIABLES

- **Marginal PMF:**
  - the PMF of each individual RV (exactly the same as in Ch. 2)
  - Joint PMF → marginal PMF (using the total probability equation)

$$\Pr(X = x_i) = \sum_j \Pr(X = x_i, Y = y_j)$$

the summation is over all the possible value of Y

$$\Pr(Y = y_j) = \sum_i \Pr(X = x_i, Y = y_j)$$

the summation is over all the possible value of X

  - Recall: total probability

$$\Pr(A) = \sum_{j=1}^{n} \Pr(A|B_j)\Pr(B_j) = \sum_{j=1}^{n} \Pr(A, B_j) \qquad B_1 \cup B_2 \cup \cdots B_n = S$$

TWO DISCRETE RV

- **Example**
  - Consider an urn contains 4 white balls and 6 black balls. Pick two balls without replacement. Let X denote the result from the first pick, and Y the result from the 2^{nd} pick, i.e., define two random variables (X, Y) as

    {(B, B)} → (X=0, Y=0), {(B, W)}→ (X=0, Y=1),

    {(W, B)}→ (X=1, Y=0), {(W, W)} →(X = 1,Y = 1)

    (a) Find the joint PMF

    (b) Find the marginal PMF of X

    (c) Find the marginal PMF of Y

$$\sum_i \sum_j P_{XY}(x_i, y_j) = 1$$

TWO DISCRETE RV

- **Review: conditional probability**
  - Two events A and B

$$P(A|B) = \frac{P(AB)}{P(B)}$$

- **Conditional PMF**
  - Given event B, what is the probability that $X = x_i$?

$$\Pr(X = x_i | B) = \frac{\Pr(X = x_i, B)}{\Pr(B)}$$

  - E.g. two RV X and Y
    - B: $Y = y_j$ $\quad \Pr(X = x_i | Y = y_j) = \frac{\Pr(X = x_i, Y = y_j)}{\Pr(Y = y_j)}$
    - B: $Y > y$ $\quad \Pr(X = x_i | Y > y) = \frac{\Pr(X = x_i, Y > y)}{\Pr(Y > y)}$

TWO DISCRETE RV

- **Example**
  - Consider an urn contains 4 white balls and 6 black balls. Pick two balls without replacement. Let X denote the result from the first pick, and Y the result from the 2nd pick, i.e., define two random variables (X, Y) as

    {(B, B)} → (X=0, Y=0), {(B, W)}→ (X=0, Y=1),

    {(W, B)}→ (X=1, Y=0), {(W, W)} →(X = 1,Y = 1)
  - Find $P(X = x_i \mid Y = y_j)$

$$\sum_i P(X = x_i \mid B) = 1$$

TWO DISCRETE RV

- **Example**
  - The joint PMF of two RV X and Y are given as follows. Find the marginal PMF and the conditional PMF $\Pr(Y=2|X=1)$

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	0.01	0	0
$x=1$	0.09	0.09	0
$x=2$	0	0	0.81

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TWO CONTINUOUS RANDOM VARIABLES

- **Joint CDF**
  - The joint cumulative distribution function (CDF) of two random variables X and Y is defined as

$$F_{XY}(x, y) = \Pr(\{X \le x\} \cap \{Y \le y\})$$

    - The above definition is true for both discrete RV and continuous RV
- **Marginal CDF**
  - The CDF of each individual RV (exactly the same as in Ch. 2)
  - joint CDF → marginal CDF

$$F_X(x) = F_{XY}(x, \infty) \qquad F_Y(y) = F_{XY}(\infty, y)$$

    - Why?

TWO CONTINUOUS RV

- **Example**

$$F_{XY}(x, -\infty) =$$

$$F_{XY}(-\infty, y) =$$

$$F_{XY}(\infty, \infty) =$$

TWO CONTINUOUS RV

- **Example**
  - Two RVs X and Y have a joint CDF as follows. Find the CDF of X

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ xy & 0 \le x \le 1, 0 \le y \le 1 \\ x & 0 \le x \le 1, y > 1 \\ y & 0 \le y \le 1, x > 1 \\ 1 & x \ge 1, y \ge 1. \end{cases}$$

TWO CONTINUOUS RV

- **Joint pdf of two continuous RVs**

$$f(x,y) = \frac{d^2 F(x,y)}{dxdy}$$

– $P[x < X \leq x + dx, y < Y \leq y + dy] = f_{X,Y}(x, y)\, dx\, dy.$

– $f_{X,Y}(x, y) \geq 0$ *for all* (x, y),

TWO CONTINUOUS RV

- **Properties of joint pdf**
  - Probability

$$P(a_1 < X < b_1, a_2 < X < b_2) = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f_{XY}(x, y)dxdy$$

  - CDF

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x, y)dxdy$$

  - Marginal pdf

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y)dy \qquad f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y)dx$$

    - Recall: marginal PMF for discrete RV

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j)$$

TWO CONTINUOUS RV

- **Example**
  - Two RVs with joint pdf given as follows

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

    - Find the marginal pdf of *X*
    - The probability that X > 0.5 and Y < 0.3

TWO CONTINUOUS RV

- **Example**
  - Two RVs with the joint pdf as follows

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

    - Find the constant c
    - Find the marginal pdf of *x* and *y*

TWO CONTINUOUS RV

- **Conditional probability**

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} \qquad f_{XY}(x, y) = f_{X|Y}(x \mid y) f_Y(y)$$

  - Recall $\quad P(A \mid B) = \dfrac{P(AB)}{P(B)} \qquad P(AB) = P(A \mid B) P(B)$

TWO CONTINUOUS RV

- **Conditional pdf**
  - Conditional pdf is still a pdf

$$0 \le f_{X|Y}(x|y) \le 1$$

$$\int_{-\infty}^{+\infty} f_{X|Y}(x|y)dx = 1$$

$$F_{X|Y}(x|y) = P(X < x|Y = y) = \int_{-\infty}^{x} f_{X|Y}(u|y)du$$

  - Difference between $P(X < x|Y = y)$ and $P(X < x|Y < y)$

$$P(X < x|Y < y) =$$

TWO CONTINUOUS RV

- **Example**
  - Two RVs with joint pdf as follows

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y} & 0 \le y \le x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

  - Find $f_{X|Y}(x|y)$
  - Find $P(X < 4 | y = 3)$
  - Find $P(X < 2 | y = 3)$

TWO CONTINUOUS RV

- **Example**
  - Two RVs with a joint pdf as follows

$$f_{XY}(x,y)=\begin{cases}\frac{6}{5}(1-x^2y), & 0\le x\le 1, 0\le x\le 1,\\ 0, & otherwise\end{cases}$$

- Find $f_{X|Y}(x|y)$
- Verify that $\int_{-\infty}^{+\infty} f_{X|Y}(x|y)dx=1$

TWO CONTINUOUS RVS

- **Joint moment**

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{XY}(x,y)\,dxdy$$

- **Linearity of the expectation operator**

$$E[aX + bY] = aE[X] + bE[Y]$$

  – Proof:

TWO CONTINUOUS RVS

- **Example**
  - Two RVs X and Y with mean $m_X = 5$ and $m_Y = 6$. Define Z = 2X + 5Y + 7. Find E[Z].

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INDEPENDENCE AND CORRELATION

- **Independence**
  - Tow continuous RVs are called independent if and only if

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

  - Recall: for two RVs X and Y

$$f_{XY}(x,y) = f_{X|Y}(x\,|\,y) f_Y(y)$$

    - Therefore, if X and Y are independent, then

$$f_{X|Y}(x\,|\,y) = f_X(x)$$

  - Recall: two events A and B are independent if and only if

$$\Pr(AB) = \Pr(A)\Pr(B)$$

INDEPENDENCE AND CORRELATION

- **Example**
  - Two RVs with joint pdf as follows

$$f_{XY}(x,y)=\begin{cases} ke^{-(x+y+1)}, & 0\le x\le\infty, 1\le y\le\infty, \\ 0, & otherwise \end{cases}$$

    - Find the value of k
    - Are they independent

INDEPENDENCE AND CORRELATION

- **Correlation**
  - The correlation between two RVs X and Y is defined as

$$E\left(XY\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xyf_{XY}(x,y)dxdy$$

- **Covariance**
  - The covariance between two RVs *X* and *Y* is defined as

$$E\left[(X-m_X)(Y-m_Y)\right]$$

    - Recall: variance of *X*: $E\left[(X-m_X)^2\right]$

$$E\left[(X-m_X)(Y-m_Y)\right]=E(XY)-m_X m_Y$$

INDEPENDENCE AND CORRELATION

- **Correlation coefficient (normalized covariance)**
  - The correlation coefficient of two RV X and Y is defined as

$$\rho = \frac{E\left[(X - m_X)(Y - m_Y)\right]}{\sigma_X \sigma_Y} = \frac{E(XY) - m_X m_Y}{\sigma_X \sigma_Y}$$

INDEPENDENCE AND CORRELATION

- **Example**
  - Consider two RVs with the joint pdf as

$$f(x,y)=\begin{cases} x+y, & 0<x<1, 0<y<1 \\ 0, & o.w. \end{cases}$$

- Find the correlation, covariance, and the correlation coefficient

INDEPENDENCE AND CORRELATION

- **Uncorrelated**
  - Two RVs X and Y are said to be uncorrelated if

$$E\left(XY\right)=E(X)E(Y)$$

  - If two RVs are uncorrelated, then the covariance is , their correlation coefficient is .

- **Independence ≠ correlation**
  - Independent → uncorrelated (it's not true the other way around)
    - If X and Y are independent

$$E\left(XY\right)=$$

  - If two RVs are uncorrelated, they might not be independent.

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INDEPENDENCE AND CORRELATION

- **Example**
  - Let Θ be uniformly distributed in $[0,2\pi]$.

    $$X = \cos(\Theta) \qquad\qquad Y = \sin(\Theta)$$

    - Are they uncorrelated?

INDEPENDENCE AND CORRELATION

- **Example**
  - Two independent RVs, X and Y, have variance $\sigma_X^2 = 4, \quad \sigma_Y^2 = 16$. Find the variance of U = 2X + 3Y.

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FUNCTIONS OF TWO RVS

- **pdf of the sum of two RVs**
  - Consider two RVs X and Y, with the joint pdf $f_{XY}(x, y)$
  - Define a new RV Z = X + Y. What is the pdf of Z?
  - Sol: Find the CDF of Z, then differentiate it with respect to z.

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}(x, z - x)dx$$

FUNCTIONS OF TWO RVS

- **Pdf of the sum of two independent RVs**
  - If X and Y are independent, then the pdf of Z = X+Y is

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}(x, z-x)dx = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)dx = f_X(z) \otimes f_Y(z)$$

    - The pdf of Z is the convolution of the pdfs of X and Y

FUNCTIONS OF TWO RVS

- **Example**
  - Two independent RVs have pdf as

$$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x}, & x > 0 \\ 0, & o.w. \end{cases} \qquad f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 y}, & y > 0 \\ 0, & o.w. \end{cases}$$

    - Find the pdf of Z = X + Y

FUNCTIONS OF TWO RVS

- **Pdf of functions of two RVs**
  - Consider two RVs, X and Y, with joint pdf $f_{XY}(x, y)$
  - Let $Z = g_1(X,Y)$, $W = g_2(X,Y)$
  - Or, inversely, $X = h_1(Z,W)$, $Y = h_2(Z,W)$
  - The joint pdf of Z and W is

$$f_{ZW}(z,w) = f_{XY}\left[h_1(z,w), h_2(z,w)\right] |J|$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix}$$

FUNCTION OF TWO RVS

- **Example**
  - Two random variables X and Y have a joint pdf

$$f_{XY}(x,y) = \begin{cases} 1, & 0 < x < 1, 0 < y < 1 \\ 0, & otherwise \end{cases}$$

  - Find the pdf of Z = XY

FUNCTIONS OF TWO RVS

- **Example**
  - A box contains resistors whose values are independent and uniformly distributed between 10 Ohm and 20 Ohm. If two resistors are selected in random and connected in series
    - The pdf of the resistance of the serial circuit.

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MOMENT GENERATING FUNCTION

- **Moment generating function (MGF)**
  - The moment generating function of a random variable X is

$$\phi_X(s) = E\left[e^{sX}\right]$$

  - Calculation of MGF for a continuous RV

$$\phi_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx$$

    - This is the Laplace transform of the pdf.

  - Calculation of MGF for a discrete RV

$$\phi_Y(s) = \sum_{y_i \in S_Y} e^{sy_i} P_Y(y_i).$$

MOMENT GENERATING FUNCTION

- **Example:**
  - Find the MGF of the exponential RV

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

MOMENT GENERATING FUNCTION

- **Example**
  - Find the MGF of a Bernoulli RV with parameter p

MOMENT GENERATING FUNCTION

- **Theorem**
  - We can calculate the n-th moment of a random variable X by using its MGF as

$$E\left[X^n\right] = \left.\frac{d^n \phi_X(s)}{ds^n}\right|_{s=0}.$$

MOMENT GENERATING FUNCTION

- **Example**
  - Find the first and second moments of an exponential RV by using the MGF

MOMENT GENERATING FUNCTION

- **MGF of sum of independent RVs**

For a set of independent random variables $X_1, X_2, \cdots, X_n$, the MGF of their sum $W = X_1 + \cdots + X_n$ is

$$\phi_W(s) = \phi_{X_1}(s)\phi_{X_2}(s)\cdots\phi_{X_n}(s)$$

MOMENT GENERATING FUNCTION

- **Example**

  For a Poisson RV with PMF

$$P_X(x) = \begin{cases} \alpha^x e^{-\alpha}/x! & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The MGF is $e^{\alpha(e^s-1)}$

– Find the distribution of the sum of N independent Poisson RVs

MOMENT GENERATING FUNCTION

- **Example**

The MGF of a Gaussian RV $X \sim \mathcal{N}(m, \sigma^2)$ is $\phi_X(s) = \exp\left(ms + \frac{\sigma^2 s^2}{2}\right)$. Find the MGF of the sum of N independent Gaussian RVs.