OUTLINE

- Two discrete random variables
- Two continuous random variables
- Statistical independence and correlation
- Functions of two random variables
- Moment generating function
TWO DISCRETE RANDOM VARIABLES

- Joint PMF of two discrete random variables
  
  Consider two discrete RVs, $X$ and $Y$. The joint probability mass function (PMF) of $X$ and $Y$ is defined as

$$p_{XY}(x_i, y_j) = \Pr(X = x_i, Y = y_j) = \Pr(\{X = x_i\} \cap \{Y = y_j\})$$
TWO DISCRETE RANDOM VARIABLES

• **Marginal PMF:**
  – the PMF of each individual RV (exactly the same as in Ch. 2)
  – Joint PMF $\rightarrow$ marginal PMF (using the total probability equation)

\[
\Pr(X = x_i) = \sum_j \Pr(X = x_i, Y = y_j)
\]

the summation is over all the possible value of $Y$

\[
\Pr(Y = y_j) = \sum_i \Pr(X = x_i, Y = y_j)
\]

the summation is over all the possible value of $X$

– Recall: total probability

\[
\Pr(A) = \sum_{j=1}^{n} \Pr(A|B_j)\Pr(B_j) = \sum_{j=1}^{n} \Pr(A, B_j)
\]

$B_1 \cup B_2 \cup \cdots B_n = S$
**Example**

- Consider an urn contains 4 white balls and 6 black balls. Pick two balls without replacement. Let $X$ denote the result from the first pick, and $Y$ the result from the 2nd pick, i.e., define two random variables $(X, Y)$ as

  \[
  \begin{align*}
  &\{(B, B)\} \rightarrow (X=0, Y=0),
  &\{(B, W)\} \rightarrow (X=0, Y=1),
  &\{(W, B)\} \rightarrow (X=1, Y=0),
  &\{(W, W)\} \rightarrow (X = 1, Y = 1)
  \end{align*}
  \]

(a) Find the joint PMF

(b) Find the marginal PMF of $X$

(c) Find the marginal PMF of $Y$

\[
\sum \sum P_{XY}(x_i, y_j) = 1
\]
TWO DISCRETE RV

- **Review: conditional probability**
  - Two events $A$ and $B$
  \[ P(A | B) = \frac{P(AB)}{P(B)} \]

- **Conditional PMF**
  - Given event $B$, what is the probability that $X = x_i$?
  \[ \Pr(X = x_i | B) = \frac{\Pr(X = x_i, B)}{\Pr(B)} \]

  - E.g. two RV $X$ and $Y$
    - B: $Y = y_j$
      \[ \Pr(X = x_i | Y = y_j) = \frac{\Pr(X = x_i, Y = y_j)}{\Pr(Y = y_j)} \]
    - B: $Y > y$
      \[ \Pr(X = x_i | Y > y) = \frac{\Pr(X = x_i, Y > y)}{\Pr(Y > y)} \]
**Example**

- Consider an urn contains 4 white balls and 6 black balls. Pick two balls without replacement. Let $X$ denote the result from the first pick, and $Y$ the result from the 2nd pick, i.e., define two random variables $(X, Y)$ as

  - $\{(B, B)\} \rightarrow (X=0, Y=0)$, $\{(B, W)\} \rightarrow (X=0, Y=1)$,
  - $\{(W, B)\} \rightarrow (X=1, Y=0)$, $\{(W, W)\} \rightarrow (X=1, Y=1)$

- Find $P(X = x_i \mid Y = y_j)$
**Example**

The joint PMF of two RV X and Y are given as follows. Find the marginal PMF and the conditional PMF $P_r(Y = 2 | X = 1)$

<table>
<thead>
<tr>
<th>$P_{X,Y}(x,y)$</th>
<th>$y = 0$</th>
<th>$y = 1$</th>
<th>$y = 2$</th>
</tr>
</thead>
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<tr>
<td>$x = 0$</td>
<td>0.01</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x = 1$</td>
<td>0.09</td>
<td>0.09</td>
<td>0</td>
</tr>
<tr>
<td>$x = 2$</td>
<td>0</td>
<td>0</td>
<td>0.81</td>
</tr>
</tbody>
</table>
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TWO CONTINUOUS RANDOM VARIABLES

- **Joint CDF**
  - The joint cumulative distribution function (CDF) of two random variables $X$ and $Y$ is defined as
    \[ F_{XY}(x, y) = \Pr(\{X \leq x\} \cap \{Y \leq y\}) \]
  - The above definition is true for both discrete RV and continuous RV

- **Marginal CDF**
  - The CDF of each individual RV (exactly the same as in Ch. 2)
  - Joint CDF $\Rightarrow$ marginal CDF
    \[
    F_X(x) = F_{XY}(x, \infty) \quad F_Y(y) = F_{XY}(\infty, y)
    \]
  - Why?
TWO CONTINUOUS RV

• Example

\[ F_{XY}(x, -\infty) = \]

\[ F_{XY}(-\infty, y) = \]

\[ F_{XY}(\infty, \infty) = \]
TWO CONTINUOUS RV

• **Example**
  
  Two RVs $X$ and $Y$ have a joint CDF as follows. Find the CDF of $X$

  \[
  F_{X,Y}(x, y) = \begin{cases} 
  0 & x < 0 \text{ or } y < 0 \\
  xy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\
  x & 0 \leq x \leq 1, y > 1 \\
  y & 0 \leq y \leq 1, x > 1 \\
  1 & x \geq 1, y \geq 1.
  \end{cases}
  \]
TWO CONTINUOUS RV

- Joint pdf of two continuous RVs

\[
f(x, y) = \frac{d^2 F(x, y)}{dxdy}
\]

\[- \quad P [x < X \leq x + dx, y < Y \leq y + dy] = f_{X,Y}(x, y) \ dx \ dy.
\]

\[- \quad f_{X,Y}(x, y) \geq 0 \ for \ all \ (x, y),
\]
TWO CONTINUOUS RV

• **Properties of joint pdf**
  
  – Probability

\[
P(a_1 < X < b_1, a_2 < X < b_2) = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f_{XY}(x, y) \, dx \, dy
\]

– CDF

\[
F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x, y) \, dx \, dy
\]

– Marginal pdf

\[
f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) \, dy
\]

\[
f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y) \, dx
\]

• Recall: marginal PMF for discrete RV

\[
P(X = x_i) = \sum_j P(X = x_i, Y = y_j)
\]
TWO CONTINUOUS RV

• **Example**
  
  – Two RVs with joint pdf given as follows

  \[
  f_{X,Y}(x, y) = \begin{cases} 
  1 & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\
  0 & \text{elsewhere.}
  \end{cases}
  \]

  • Find the marginal pdf of \( X \)

  • The probability that \( X > 0.5 \) and \( Y < 0.3 \)
TWO CONTINUOUS RV

- **Example**
  - Two RVs with the joint pdf as follows
    \[
    f_{X,Y}(x, y) = \begin{cases} 
    ce^{-x}e^{-y} & 0 \leq y \leq x < \infty \\
    0 & \text{elsewhere.}
    \end{cases}
    \]
  - Find the constant c
  - Find the marginal pdf of x and y
TWO CONTINUOUS RV

- **Conditional probability**

\[ f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} \]

\[ f_{XY}(x, y) = f_{X|Y}(x \mid y)f_Y(y) \]

- Recall

\[ P(A \mid B) = \frac{P(AB)}{P(B)} \]

\[ P(AB) = P(A \mid B)P(B) \]
**TWO CONTINUOUS RV**

- **Conditional pdf**
  - Conditional pdf is still a pdf

\[ 0 \leq f_{X|Y}(x \mid y) \leq 1 \]

\[ \int_{-\infty}^{+\infty} f_{X|Y}(x \mid y) \, dx = 1 \]

\[ F_{X|Y}(x \mid y) = P(X < x \mid Y = y) = \int_{-\infty}^{x} f_{X|Y}(u \mid y) \, du \]

- Difference between \( P(X < x \mid Y = y) \) and \( P(X < x \mid Y < y) \)

\[ P(X < x \mid Y < y) = \]
**Example**

- Two RVs with joint pdf as follows

\[ f_{X,Y}(x, y) = \begin{cases} 
  ce^{-x}e^{-y} & 0 \leq y \leq x < \infty \\
  0 & \text{elsewhere.}
\end{cases} \]

- Find \( f_{X|Y}(x \mid y) \)
- Find \( P(X < 4 \mid y = 3) \)
- Find \( P(X < 2 \mid y = 3) \)
• **Example**
  
  – Two RVs with a joint pdf as follows

\[
f_{XY}(x, y) = \begin{cases} 
  \frac{6}{5} (1 - x^2 y), & 0 \leq x \leq 1, 0 \leq y \leq 1, \\
  0, & \text{otherwise}
\end{cases}
\]

• Find \( f_{X|Y}(x \mid y) \)
• Verify that \( \int_{-\infty}^{+\infty} f_{X|Y}(x \mid y) \, dx = 1 \)
TWO CONTINUOUS RVS

• **Joint moment**

\[ E[g(X,Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) f_{XY}(x, y) \, dx \, dy \]

• **Linearity of the expectation operator**

\[ E[aX + bY] = aE[X] + bE[Y] \]

– Proof:
Two Continuous RVs

- Example
  - Two RVs X and Y with mean $m_X = 5$ and $m_Y = 6$. Define $Z = 2X + 5Y + 7$. Find $E[Z]$. 
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INDEPENDENCE AND CORRELATION

• Independence
  – Tow continuous RVs are called independent if and only if

\[
f_{XY}(x, y) = f_X(x)f_Y(y)
\]

– Recall: for two RVs X and Y

\[
f_{XY}(x, y) = f_{X|Y}(x \mid y)f_Y(y)
\]

• Therefore, if X and Y are independent, then

\[
f_{X|Y}(x \mid y) = f_X(x)
\]

– Recall: two events A and B are independent if and only if

\[
\Pr(AB) = \Pr(A)\Pr(B)
\]
INDEPENDENCE AND CORRELATION

• Example
  – Two RVs with joint pdf as follows
    \[ f_{XY}(x, y) = \begin{cases} 
      ke^{-(x+y+1)}, & 0 \leq x \leq \infty, 1 \leq y \leq \infty, \\
      0, & \text{otherwise} 
    \end{cases} \]
  
  • Find the value of \( k \)
  • Are they independent
INDEPENDENCE AND CORRELATION

• **Correlation**
  – The correlation between two RVs X and Y is defined as

  \[E(XY) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf_{XY}(x, y) dxdy\]

• **Covariance**
  – The covariance between two RVs X and Y is defined as

  \[E[(X - m_X)(Y - m_Y)]\]

  • Recall: variance of X: \[E[(X - m_X)^2]\]

  \[E[(X - m_X)(Y - m_Y)] = E(XY) - m_X m_Y\]
INDEPENDENCE AND CORRELATION

- Correlation coefficient (normalized covariance)
  - The correlation coefficient of two RV $X$ and $Y$ is defined as
    \[
    \rho = \frac{E[(X - m_X)(Y - m_Y)]}{\sigma_X \sigma_Y} = \frac{E(XY) - m_X m_Y}{\sigma_X \sigma_Y}
    \]
INDEPENDENCE AND CORRELATION

• Example

  – Consider two RVs with the joint pdf as

  \[
  f(x, y) = \begin{cases} 
  x + y, & 0 < x < 1, 0 < y < 1 \\
  0, & o.w.
  \end{cases}
  \]

  • Find the correlation, covariance, and the correlation coefficient
INDEPENDENCE AND CORRELATION

• **Uncorrelated**
  
  – Two RVs X and Y are said to be uncorrelated if
    \[ E(XY) = E(X)E(Y) \]
  
  – If two RVs are uncorrelated, then the covariance is 0, their correlation coefficient is 0.

• **Independence ≠ correlation**
  
  – Independent \( \rightarrow \) uncorrelated (it’s not true the other way around)
    
    • If X and Y are independent
      \[ E(XY) = \]

  – If two RVs are uncorrelated, they might not be independent.
• **Example**
  
  – Let $\Theta$ be uniformly distributed in $[0,2\pi]$. 
  
  $$X = \cos(\Theta) \quad Y = \sin(\Theta)$$

• Are they uncorrelated?
INDEPENDENCE AND CORRELATION

- Example
  - Two independent RVs, X and Y, have variance $\sigma_X^2 = 4$, $\sigma_Y^2 = 16$. Find the variance of $U = 2X + 3Y$. 

---

\[ \sigma_X^2 = 4, \quad \sigma_Y^2 = 16 \]
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FUNCTIONS OF TWO RVS

• pdf of the sum of two RVs
  – Consider two RVs X and Y, with the joint pdf \( f_{XY}(x, y) \)
  – Define a new RV \( Z = X + Y \). What is the pdf of \( Z \)?
  – Sol: Find the CDF of \( Z \), then differentiate it with respect to \( z \).

\[
f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}(x, z-x)\,dx
\]
FUNCTIONS OF TWO RVS

• **Pdf of the sum of two independent RVs**
  – If X and Y are independent, then the pdf of \( Z = X+Y \) is

\[
f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}(x, z-x)dx = \int_{-\infty}^{+\infty} f_X(x)f_Y(z-x)dx = f_X(z) \otimes f_Y(z)
\]

• The pdf of Z is the convolution of the pdfs of X and Y
FUNCTIONS OF TWO RVS

• Example
  
  - Two independent RVs have pdf as
    
    \[ f_x(x) = \begin{cases} 
    \lambda_1 e^{-\lambda_1 x}, & x > 0 \\
    0, & o.w.
    \end{cases} \quad f_y(y) = \begin{cases} 
    \lambda_2 e^{-\lambda_2 y}, & y > 0 \\
    0, & o.w.
    \end{cases} \]
  
  - Find the pdf of \( Z = X + Y \)
FUNCTIONS OF TWO RVS

• Pdf of functions of two RVs
  – Consider two RVs, X and Y, with joint pdf \( f_{XY}(x, y) \)
  – Let \( Z = g_1(X, Y) \), \( W = g_2(X, Y) \)
  – Or, inversely, \( X = h_1(Z, W) \), \( Y = h_2(Z, W) \)
  – The joint pdf of \( Z \) and \( W \) is

\[
f_{ZW}(z, w) = f_{XY}(h_1(z, w), h_2(z, w)) |J|
\]

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial x} & \frac{\partial x}{\partial w} \\
\frac{\partial z}{\partial x} & \frac{\partial z}{\partial w} \\
\frac{\partial y}{\partial x} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial y} & \frac{\partial z}{\partial w}
\end{vmatrix}
\]
FUNCTION OF TWO RVS

• **Example**
  - Two random variables $X$ and $Y$ have a joint pdf
    
    $$f_{XY}(x, y) = \begin{cases} 
    1, & 0 < x < 1, 0 < y < 1 \\
    0, & \text{otherwise} 
    \end{cases}$$
  
  - Find the pdf of $Z = XY$
FUNCTIONS OF TWO RVS

• Example
  – A box contains resistors whose values are independent and uniformly distributed between 10 Ohm and 20 Ohm. If two resistors are selected in random and connected in series
    • The pdf of the resistance of the serial circuit.
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MOMENT GENERATING FUNCTION

- **Moment generating function (MGF)**
  - The moment generating function of a random variable $X$ is

$$\phi_X(s) = E \left[ e^{sX} \right]$$

- Calculation of MGF for a continuous RV

$$\phi_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) \, dx$$

- This is the Laplace transform of the pdf.

- Calculation of MGF for a discrete RV

$$\phi_Y(s) = \sum_{y_i \in S_Y} e^{sy_i} P_Y(y_i).$$
MOMENT GENERATING FUNCTION

- **Example:**
  - Find the MGF of the exponential RV

\[
f_X(x) = \begin{cases} 
  \lambda e^{-\lambda x} & x \geq 0 \\
  0 & \text{otherwise} 
\end{cases}
\]
MOMENT GENERATING FUNCTION

- **Example**
  - Find the MGF of a Bernoulli RV with parameter $p$
MOMENT GENERATING FUNCTION

• **Theorem**
  
  We can calculate the n-th moment of a random variable $X$ by using its MGF as

  $$E\left[X^n\right] = \frac{d^n \phi_X(s)}{ds^n}\bigg|_{s=0}.$$
• **Example**
  – Find the first and second moments of an exponential RV by using the MGF
MOMENT GENERATING FUNCTION

• MGF of sum of independent RVs

For a set of independent random variables $X_1, X_2, \cdots, X_n$, the MGF of their sum $W = X_1 + \cdots + X_n$ is

$$
\phi_W(s) = \phi_{X_1}(s)\phi_{X_2}(s) \cdots \phi_{X_n}(s)
$$
MOMENT GENERATING FUNCTION

- **Example**

  For a Poisson RV with PMF

  \[ P_X(x) = \begin{cases} \alpha^x e^{-\alpha} / x! & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \]

  The MGF is \( e^{\alpha (e^s - 1)} \)

  - Find the distribution of the sum of N independent Poisson RVs
• Example

The MGF of a Gaussian RV $X \sim \mathcal{N}(m, \sigma^2)$ is $\phi_X(s) = \exp \left( ms + \frac{\sigma^2 s^2}{2} \right)$. Find the MGF of the sum of $N$ independent Gaussian RVs.