Department of Electrical Engineering
University of Arkansas

ELEG 3143 Probability & Stochastic Process
Ch. 3 Continuous Random Variables

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OUTLINE

- **Continuous Random Variable**
- **Family of continuous RV**
- **Moments of Continuous RV**
- **Gaussian RV**
- **Functions of one RV**

CONTINUOUS RV

- **Continuous RV**
  - If a random variable can take an unaccountable number of values, then the random variable is a continuous random variable.
  - Examples of continuous RV
    - The daily average temperature
    - The expected lifetime of a computer
    - The amplitude of noise in an electronic component
    - ……

CONTINUOUS RV

- **Cumulative distribution function (distribution function)**
  - The cumulative distribution function (CDF) of a continuous RV is

$$F_X(x) = \Pr(X \le x)$$

    - The probability that the RV X is smaller than or equal to x
  - Recall: the CDF of a discrete RV Y is $F_Y(y) = \Pr(Y \le y)$

CONTINUOUS RV

- **Properties of CDF**

$$0 \le F_X(x) \le 1$$

$$F_X(-\infty) = 0$$

$$F_X(\infty) = 1$$

$$\Pr(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$$

$F_X(x)$ is a non-decreasing function of x

CONTINUOUS RV

- **Example**
  - A particular random variable has a probability distribution function given by

$$F_X(x) = \begin{cases} 1 - e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

    - What is the probability that $X \leq 0.25$
    - What is the probability that $X > 0.5$
    - What is the probability that $0.25 < X \leq 0.5$

CONTINUOUS RV

- **Probability density function (pdf)**

$$f(x) = \frac{dF(x)}{dx}$$

  – The "density" of probability

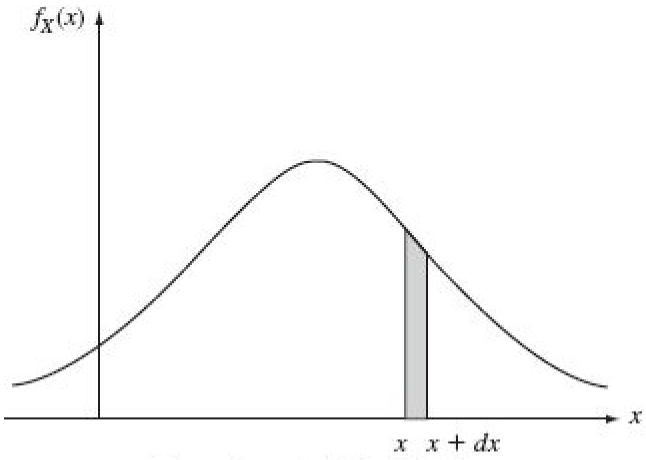
  – Review: differentiation

$$\frac{dF(x)}{dx} = \lim_{\Delta \to 0} \frac{F(x+\Delta) - F(x)}{\Delta}$$

  – Interpretation:

$$f(x) \approx \frac{P(x < X \le x + \Delta)}{\Delta}$$

$$P\{x < X \le x + \Delta\} = f(x)\Delta$$



CONTINUOUS RV

- **Relationship between pdf and CDF**

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)dy$$

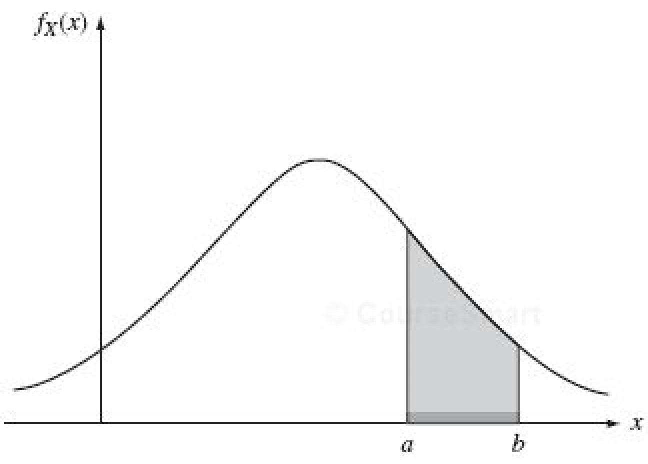
– $\Pr(X \le a) = F_X(a) = \int_{-\infty}^{a} f_X(y)dy$

$\Pr(X \le b) = F_X(b) = \int_{-\infty}^{b} f_X(y)dy$

$$\Pr(a < X \le b) = F_X(b) - F_X(a) = \int_{a}^{b} f_X(y)dy$$

– Review: integration
  - finding the area under the integrand

$$\int_{a}^{b} f_X(y)dy \approx \sum f_X(y_i)\Delta$$



CONTINUOUS RV

- **Properties of pdf**

$$f(x) \geq 0$$

$$P\{a < X \leq b\} = \int_a^b f(x)dx$$

$$\int_{-\infty}^{b} f(x)dx = F(b)$$

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

CONTINUOUS RV

- **Example**
  - The pdf of a RV X has the form $f(x) = 5e^{-ax}u(x)$
    - The value of a
    - the probability that $X \leq 2$
    - The probability that $1 < X \leq 2$

OUTLINE

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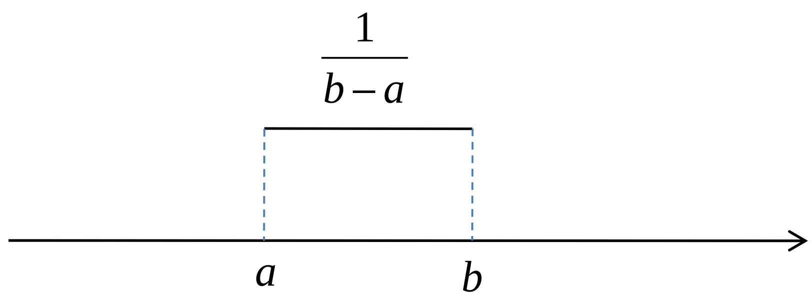
CONTINUOUS RV

- **Uniform distribution**
  - A random variable X is said to be uniformly distributed over the interval (a, b) if its pdf is given by

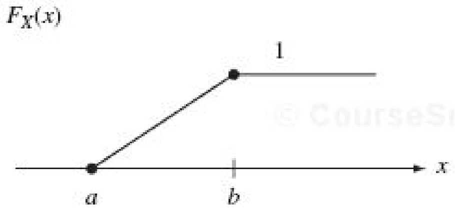
$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a < x \le b \\ 0, & \text{otherwise} \end{cases}$$

  - The RV has equal probability of being any number inside (a, b)



CONTINUOUS RV

- **Uniform distribution**
  - CDF of a uniform RV



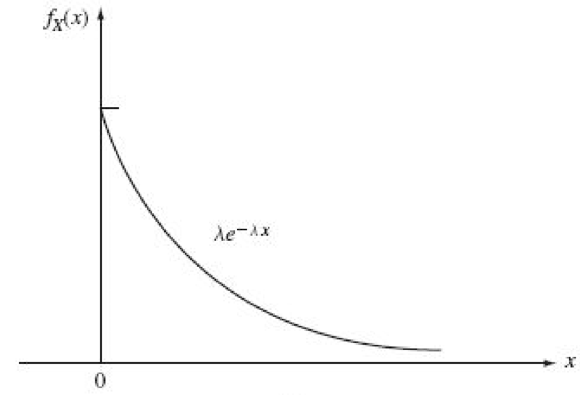
CONTINUOUS RV

- **Example**
  - A continuous RV X is uniformly distributed between (3, 8). Find the following probability
    - $\Pr(4 < X \leq 5)$
    - $\Pr(2 < X \leq 5)$
    - $\Pr(2 < X \leq 9)$

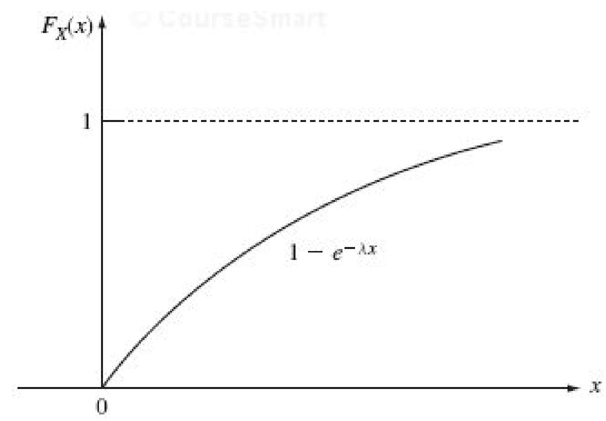
CONTINUOUS RV

- **Exponential RV**
  - The pdf of an exponential RV with parameter $\lambda > 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



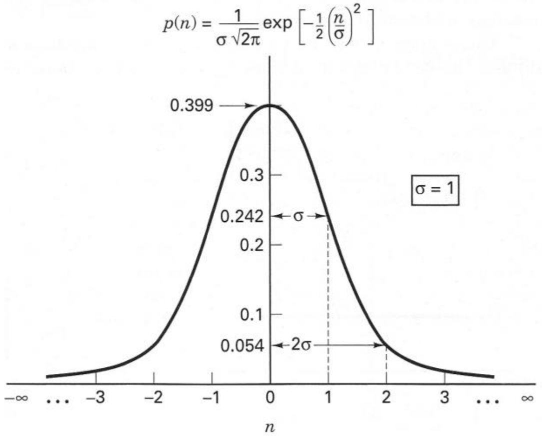
  - The CDF of an exponential RV



CONTINUOUS RV

- **Normal RV (Gaussian RV)**
  - A Gaussian RV X with parameters m and σ^2

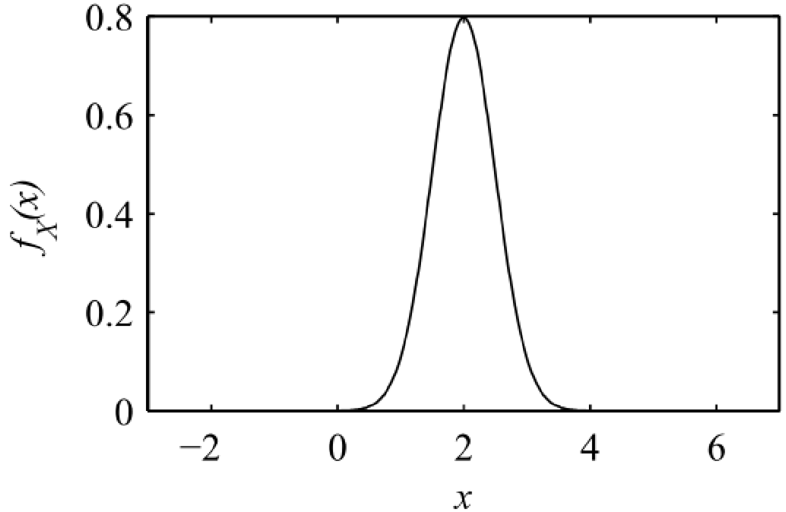
$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left[-\frac{(x-m)^2}{2\sigma^2} \right]$$



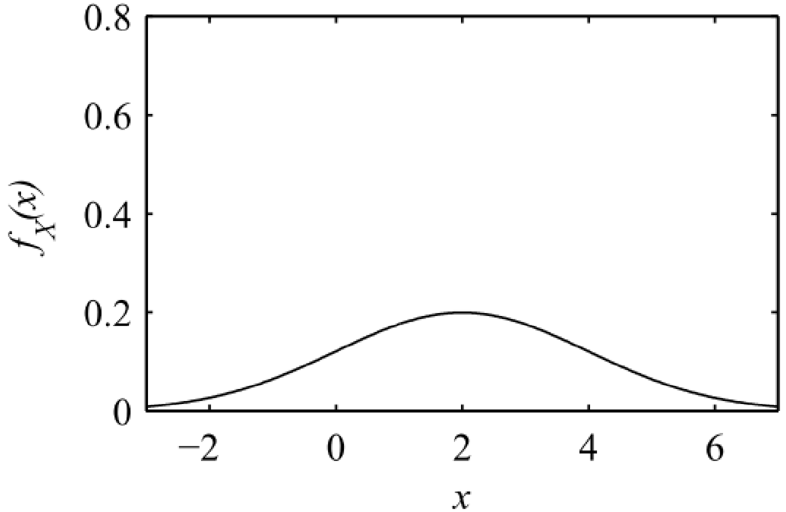
CONTINUOUS RV

- **Normal RV (Gaussian RV)**

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$



$m_X = 2, \sigma = 1/2$



$m_X = 2, \sigma = 2$

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MOMENTS: DISCRETE RV

- **The expected value, or mean, of a continuous RV, *X*, is**

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx$$

  - Recall: discrete RV $E(X) = \sum_{x_i} x_i p_X(x_i)$
  - Also known as: expected value, mean, average, first moment
  - Interpretation
    - The *weighted average* of the random variable
    - Recall: integration is an extreme case of summation.
  - Also denoted as

$$E(X) = \overline{X} = m_X$$

MOMENTS: CONTINOUS RV

- **Example**
  - The mean of an RV X uniformly distributed in (a, b)

MOMENTS: CONTINOUS RV

- **Example**
  - The mean of an exponential RV with parameter λ

MOMENTS: CONTINUOUS RV

- **Expected value of any function of X**

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

  - Recall: discrete RV $E[g(X)] = \sum_{x_i} g(x_i) p_X(x_i)$
  - $E(X)$: the expectation operator
  - The variable of the expectation operator must be a random variable
    - X is an RV
    - g(X) is an RV
  - Also denoted as

$$E[g(X)] = \overline{g(X)}$$

MOMENTS: CONTINUOUS RV

- **Variance**
  - The variance of a continuous RV X is

$$\text{Var}(X) = \sigma_X^2 = E\left[(X - m_X)^2\right] = \int_{-\infty}^{+\infty} (x - m_X)^2 f_X(x)dx$$

  - Recall: discrete RV $\sigma_X^2 = \sum_{x_i} (x_i - m_X)^2 p_X(x_i)$
  - Also called: the 2nd central moment of *X*
  - Standard deviation: σ_X
  - Physical interpretation
    - How far away the random variable is from its expected value

MOMENTS: CONTINUOUS RV

- **Properties of variance**
  - $$\sigma_X^2 = E[X^2] - m_X^2$$
  - $$\text{Var}[aX + b] = a^2\text{Var}[X]$$

MOMENTS: CONTINUOUS RV

- **The *n*-th moment of the random variable X**

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f_X(x)dx$$

  - 1st moment: $E[X]$ mean
  - 2nd moment: $E[X^2]$

- **The n-th central moment of the random variable X**

$$E[(X - m_X)^n] = \int_{-\infty}^{+\infty} (x - m_X)^n f_X(x)dx$$

  - 1st central moment: $E[X - m_X] =$
  - 2nd central moment: $E\left[(X - m_X)^2\right] = \sigma_X^2$ variance

- **The combination of all the moments, n = 1, 2, …., gives a complete description of the random variable**

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MOMENTS: CONTINUOUS RV

- **Example**
  - Find the variance of an RV X uniformly distributed on (a, b)

MOMENTS

- **Example**
  - Consider a random variable $Y = A\cos(2\pi f + \Theta)$, where A, f are constants, and Θ is uniformly distributed in $[0,2\pi]$. Find the expected value of *Y*.

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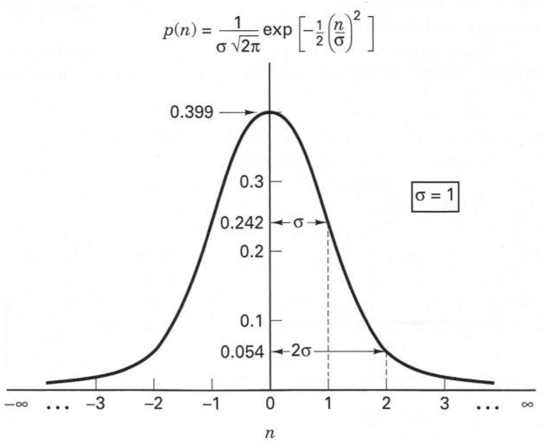
GAUSSIAN RV

- **Normal RV (Gaussian RV)**

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left[-\frac{(x-m)^2}{2\sigma^2} \right]$$



– The mean and variance are

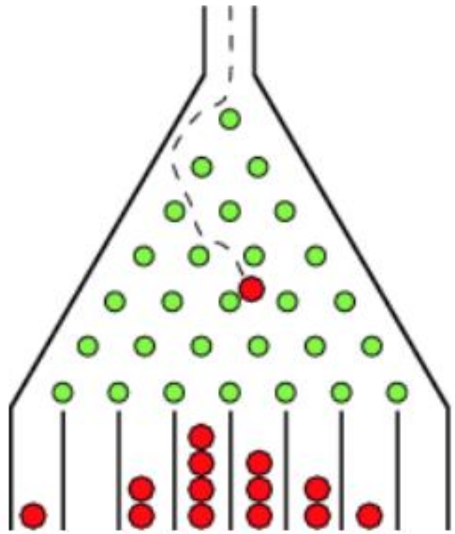
$$\mathbb{E}[X] = m$$

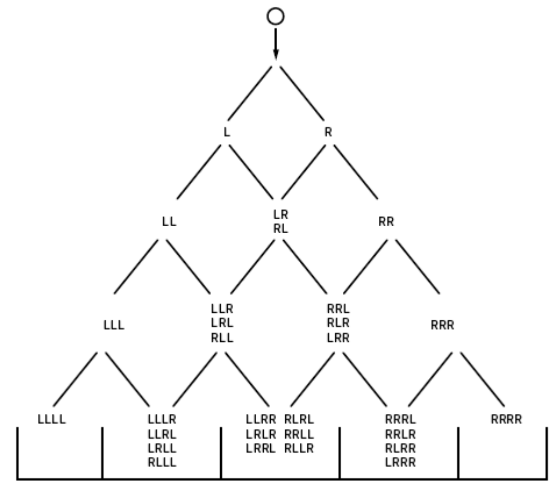
$$\mathrm{Var}[X] = \sigma^2$$

– The Gaussian RV is completely determined by its first 2 moments.
  – Not true for other RVs.

GAUSSIAN RV

- **Galton board**



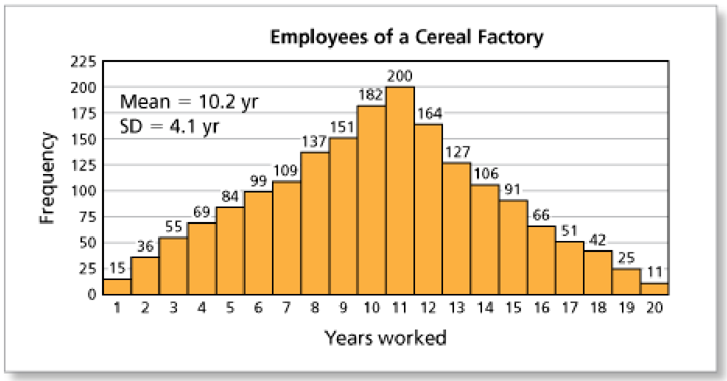


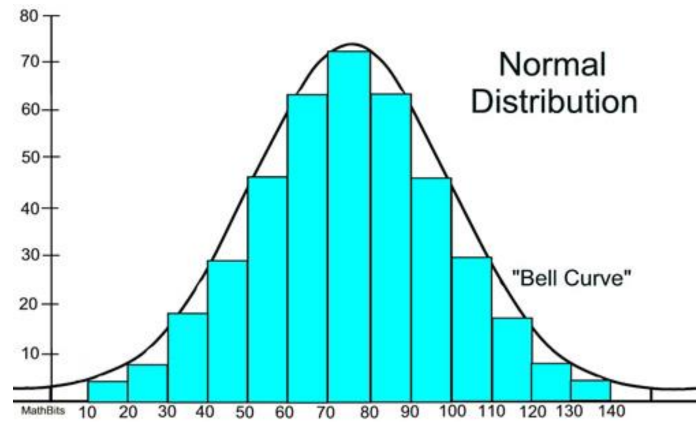
  – A video demonstration

    https://www.youtube.com/watch?v=xDIyAOBa_yU

GAUSSIAN RV

- **Probability density function (pdf) v.s. Histogram**
  - years worked of 1820 employees in a cereal factory





  - When the bin width goes to 0, histogram ➔ pdf

GAUSSIAN RV

- **Notation of Gaussian RV**
  - A Gaussian RV X with mean m and variance σ^2 is denoted as

$$X \sim \mathcal{N}(m, \sigma^2)$$

- **Standard Gaussian RV**
  - A Gaussian RV with

$$m = 0, \sigma^2 = 1$$

  - Denoted as

$$X \sim \mathcal{N}(0, 1)$$

GAUSSIAN RV

- **A linear transformation of a Gaussian RV is still Gaussian**
  - Consider $X \sim \mathcal{N}(m, \sigma^2)$. Define a new RV

$$Y = aX + b$$

  - Then Y is still Gaussian distributed

$$Y \sim \mathcal{N}(am + b, a^2\sigma^2)$$

- **Converting a Gaussian RV to a standard Gaussian RV**
  - If

$$X \sim \mathcal{N}(m, \sigma^2)$$

  - Then

$$\frac{X - m}{\sigma} \sim \mathcal{N}(0, 1)$$

GAUSSIAN RV

- **CDF of standard Gaussian random variable**
  - Let $Z \sim \mathcal{N}(0,1)$ be a standard Gaussian random variable, then the CDF of *Z* is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$

- **Complementary CDF (CCDF) of standard Gaussian random variable**
  - Let $Z \sim \mathcal{N}(0,1)$

$$Q(z) = P\left[Z > z\right] = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-u^2/2}\, du = 1 - \Phi(z).$$

GAUSSIAN RV

- **Q-function table**

$$Q(z) = \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

Table 1: Values of $Q(x)$ for $0 \le x \le 9$

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.00	0.5	2.30	0.010724	4.55	2.6823×10^{-6}	6.80	5.231×10^{-12}
0.05	0.48006	2.35	0.0093867	4.60	2.1125×10^{-6}	6.85	3.6925×10^{-12}
0.10	0.46017	2.40	0.0081975	4.65	1.6597×10^{-6}	6.90	2.6001×10^{-12}
0.15	0.44038	2.45	0.0071428	4.70	1.3008×10^{-6}	6.95	1.8264×10^{-12}
0.20	0.42074	2.50	0.0062097	4.75	1.0171×10^{-6}	7.00	1.2798×10^{-12}
0.25	0.40129	2.55	0.0053861	4.80	7.9333×10^{-7}	7.05	8.9459×10^{-13}
0.30	0.38209	2.60	0.0046612	4.85	6.1731×10^{-7}	7.10	6.2378×10^{-13}
0.35	0.36317	2.65	0.0040246	4.90	4.7918×10^{-7}	7.15	4.3389×10^{-13}
0.40	0.34458	2.70	0.003467	4.95	3.7107×10^{-7}	7.20	3.0106×10^{-13}
0.45	0.32636	2.75	0.0029798	5.00	2.8665×10^{-7}	7.25	2.0839×10^{-13}
0.50	0.30854	2.80	0.0025551	5.05	2.2091×10^{-7}	7.30	1.4388×10^{-13}
0.55	0.29116	2.85	0.002186	5.10	1.6983×10^{-7}	7.35	9.9103×10^{-14}
0.60	0.27425	2.90	0.0018658	5.15	1.3024×10^{-7}	7.40	6.8092×10^{-14}
0.65	0.25785	2.95	0.0015889	5.20	9.9644×10^{-8}	7.45	4.667×10^{-14}
0.70	0.24196	3.00	0.0013499	5.25	7.605×10^{-8}	7.50	3.1909×10^{-14}
0.75	0.22663	3.05	0.0011442	5.30	5.7901×10^{-8}	7.55	2.1763×10^{-14}
0.80	0.21186	3.10	0.0009676	5.35	4.3977×10^{-8}	7.60	1.4807×10^{-14}
0.85	0.19766	3.15	0.00081635	5.40	3.332×10^{-8}	7.65	1.0049×10^{-14}
0.90	0.18406	3.20	0.00068714	5.45	2.5185×10^{-8}	7.70	6.8033×10^{-15}
0.95	0.17106	3.25	0.00057703	5.50	1.899×10^{-8}	7.75	4.5946×10^{-15}
1.00	0.15866	3.30	0.00048342	5.55	1.4283×10^{-8}	7.80	3.0954×10^{-15}
1.05	0.14686	3.35	0.00040406	5.60	1.0718×10^{-8}	7.85	2.0802×10^{-15}
1.10	0.13567	3.40	0.00033693	5.65	8.0224×10^{-9}	7.90	1.3945×10^{-15}
1.15	0.12507	3.45	0.00028029	5.70	5.9904×10^{-9}	7.95	9.3256×10^{-16}
1.20	0.11507	3.50	0.00023263	5.75	4.4622×10^{-9}	8.00	6.221×10^{-16}
1.25	0.10565	3.55	0.00019262	5.80	3.3157×10^{-9}	8.05	4.1397×10^{-16}
1.30	0.0968	3.60	0.00015911	5.85	2.4579×10^{-9}	8.10	2.748×10^{-16}
1.35	0.088508	3.65	0.00013112	5.90	1.8175×10^{-9}	8.15	1.8196×10^{-16}
1.40	0.080757	3.70	0.0001078	5.95	1.3407×10^{-9}	8.20	1.2019×10^{-16}
1.45	0.073529	3.75	8.8417×10^{-5}	6.00	9.8659×10^{-10}	8.25	7.9197×10^{-17}
1.50	0.066807	3.80	7.2348×10^{-5}	6.05	7.2423×10^{-10}	8.30	5.2056×10^{-17}
1.55	0.060571	3.85	5.9059×10^{-5}	6.10	5.3034×10^{-10}	8.35	3.4131×10^{-17}
1.60	0.054799	3.90	4.8096×10^{-5}	6.15	3.8741×10^{-10}	8.40	2.2324×10^{-17}
1.65	0.049471	3.95	3.9076×10^{-5}	6.20	2.8232×10^{-10}	8.45	1.4565×10^{-17}
1.70	0.044565	4.00	3.1671×10^{-5}	6.25	2.0523×10^{-10}	8.50	9.4795×10^{-18}
1.75	0.040059	4.05	2.5609×10^{-5}	6.30	1.4882×10^{-10}	8.55	6.1544×10^{-18}
1.80	0.03593	4.10	2.0658×10^{-5}	6.35	1.0766×10^{-10}	8.60	3.9858×10^{-18}
1.85	0.032157	4.15	1.6624×10^{-5}	6.40	7.7688×10^{-11}	8.65	2.575×10^{-18}
1.90	0.028717	4.20	1.3346×10^{-5}	6.45	5.5925×10^{-11}	8.70	1.6594×10^{-18}
1.95	0.025588	4.25	1.0689×10^{-5}	6.50	4.016×10^{-11}	8.75	1.0668×10^{-18}
2.00	0.02275	4.30	8.5399×10^{-6}	6.55	2.8769×10^{-11}	8.80	6.8408×10^{-19}
2.05	0.020182	4.35	6.8069×10^{-6}	6.60	2.0558×10^{-11}	8.85	4.376×10^{-19}
2.10	0.017864	4.40	5.4125×10^{-6}	6.65	1.4655×10^{-11}	8.90	2.7923×10^{-19}
2.15	0.015778	4.45	4.2935×10^{-6}	6.70	1.0421×10^{-11}	8.95	1.7774×10^{-19}
2.20	0.013903	4.50	3.3977×10^{-6}	6.75	7.3923×10^{-12}	9.00	1.1286×10^{-19}
2.25	0.012224						

GAUSSIAN RV

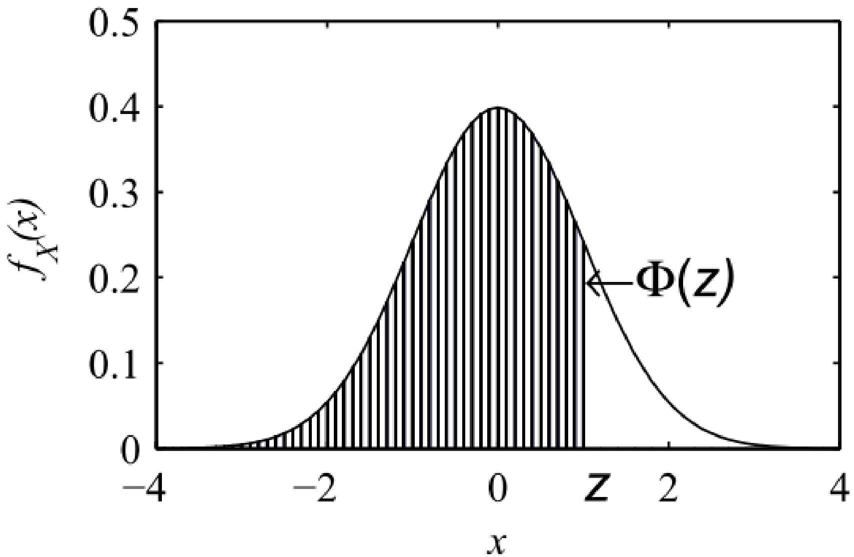
- **CDF of Gaussian Random Variable**
  - Let $X \sim \mathcal{N}(m, \sigma^2)$, then the CDF of *X* is

$$F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right)$$

GAUSSIAN RV

- **Symmetric property of the Gaussian-Q function** $\Phi(z)$

$$\Phi(-z) = 1 - \Phi(z).$$



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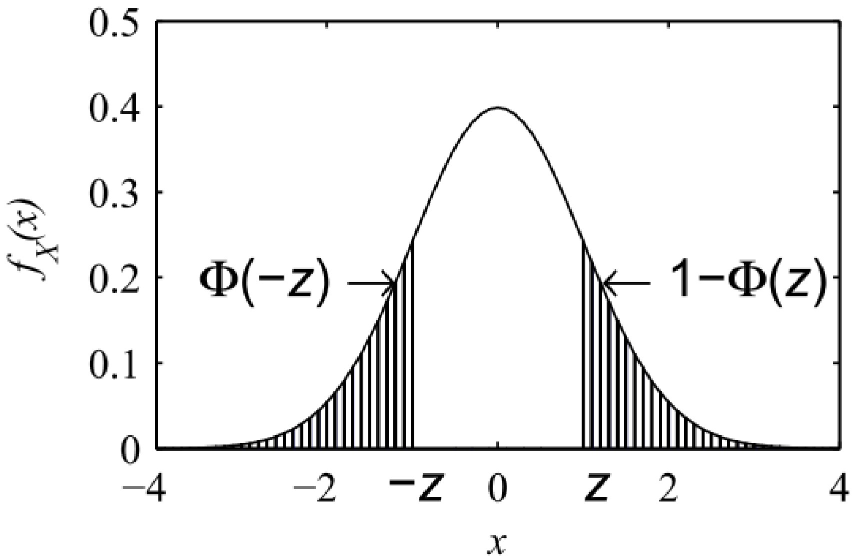

(b)

GAUSSIAN RV

- **Example**
  - Consider $X \sim \mathcal{N}(m, \sigma^2)$, what are the probability for the following events
    - $X \in [m - \sigma, m + \sigma]$
    - $X \in [m - 2\sigma, m + 2\sigma]$
    - $X \in [m - 3\sigma, m + 3\sigma]$

$$\Phi(1) = 0.8413, \quad \Phi(2) = 0.97725, \quad \Phi(3) = 1.35 \times 10^{-3}$$

GAUSSIAN RV

- **Example**
  - Consider a random variable $X \sim \mathcal{N}(1, 4)$. Find the following probabilities
    - $\Pr(X \leq -1)$
    - $\Pr(X > 3)$
    - $\Pr(-1 < X \leq 3)$

GAUSSIAN RV

- **Example**
  - In a digital communication system the probability of a binary error is $P_e = Q(\sqrt{\gamma})$, where γ is the signal-to-noise ratio (SNR). If we want $P_e \leq 10^{-4}$, what is the minimum γ?

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FUNCTIONS OF ONE RV

- **Example**
  - X is an RV with pdf $f_X(x)$. Let Y = a X + b, where a and b are constants. What is the pdf of Y?

FUNCTIONS OF ONE RV

- **Example**
  - If X is a Gaussian RV with mean 0 and variance 1. Find the pdf of

$$Y = \sigma X + m$$

FUNCTIONS OF ONE RV

- **Example**
  - If X is a Gaussian RV with mean 0 and variance 1. Find the pdf of $Y = X^2$

FUNCTIONS OF ONE RV

- **Example**
  - Let X be an exponential RV with parameter $\lambda = 2$. Find the pdf of $Y = X^2$