Department of Electrical Engineering
University of Arkansas

ELEG 3143 Probability & Stochastic Process
Ch. 1 Probability

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OUTLINE

- **Applications**
- **Elementary Set Theory**
- **Random Experiments**
- **Probability**
- **Conditional probability**
- **Independence**
- **Bayes' Formula**
- **Bernoulli trials**

APPLICARTIONS

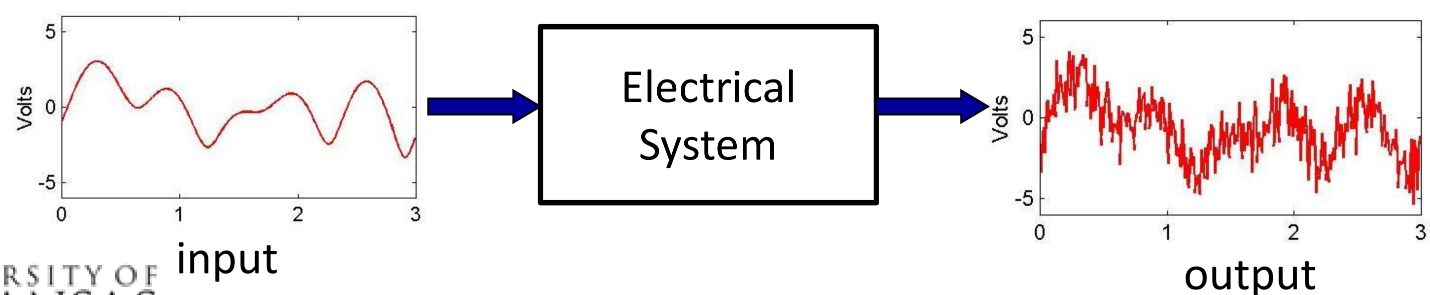
- **Why probability?**
  - Most real world events have uncertain (or random) outcomes
    - flipping a coin
    - Tossing dices
    - Football games
    - Blackjack
    - Expected lifetime of iPhone
    - The actual resistance of a 100 Ohm resistor
    - ……
  - How do we characterize these random events?
  - Can we predict the outcome of a random event?

APPLICATIONS

- **Probability Theory**
  - provides a complete set of mathematical tools and theories that can accurately and precisely describe the statistical behaviors of the random phenomenon.
  - It is a branch of Mathematics
  - It has a wide rang of applications to Engineers.

APPLICATIONS

- **Random input signals**
  - The input for a system might be random
    - E.g. the number of vehicles passing through a certain check points as an input to a traffic control system (e.g. duration of traffic lights)
    - E.g. temperature fluctuations as an input to a heating system
    - E.g. Music signals as an input to your stereo system
- **Random system characteristics**
  - The system itself has random characteristics
    - E.g. The components inside a system has random values
      - A 100 Ohm resistor might have an actual value of 101 Ohm
    - E.g. Noise: random electrical disturbance caused by the random movement of electronics
      - It is present at all electrical systems



APPLICATIONS

- **System reliability**
  - What is the expected life cycle of a given system?
  - What is the probability of failure of a system?
    - Warranty duration, insurance policy, etc.
- **Random sampling**
  - It might be too costly to inspect every single elements
  - Only sample a small population, then deduce the general behavior from the sample results
    - E.g. survey
    - E.g. product inspection for quality control
  - How to design the random sampling process?
- **Computer simulation**
  - A cost effective and efficient way to test the performance of a system
  - Random inputs, random system parameters (e.g. random component value, random disturbance, …)

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SET THEORY

- **Set**
  - A collection of elements
  - Example: $E = \{\alpha_1, \alpha_2, \Lambda\ , \alpha_n\}$
    - E is a set
    - α_m is an element of the set E
  - A subset of E is any set all of whose elements are also elements of E
    - Example: $E = \{1,2,3,4,5,6\}$
    - $F = \{1,3,6\}$ is a subset of E, $F \subset E$
    - $G = \{1\}$ is a subset of E, $G \subset E$
    - Any set is a subset of itself $E \subset E$
  - Empty set: ϕ
    - A set that does not have any element
    - Empty set is a subset of any set $\phi \subset E$
  - Space S: the largest set of interests for a given application
    - E.g. toss a coin, S ={H, T}
    - E.g. throw a die, S = {1, 2, 3, 4, 5, 6}

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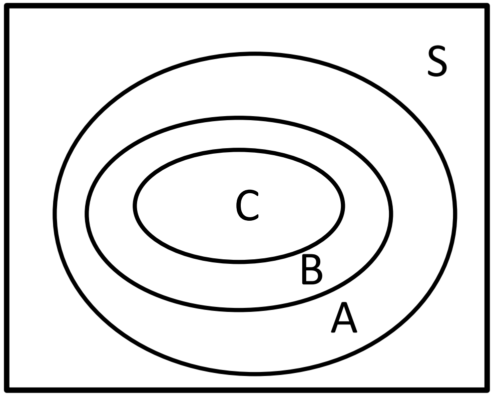
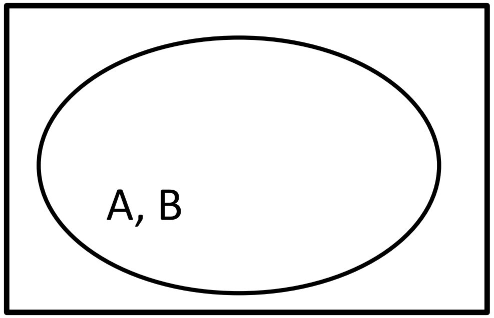
SET THEORY: VENN DIAGRAM

- **Venn diagram**
  - A geometric representation, where the space S is represented by a square and the sets are represented by closed plane figures inside the square.
  - E.g. $C \subset B \subset A$



  - If $A \subset B$ and $B \subset A$, then $A = B$



SET THEORY

- **Union (sum)**
  - The union of two sets A and B is a set consisting of all the elements from A or B or both
  - Denoted as $A \cup B$
  - Associate law

    $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$
  - Commutative law

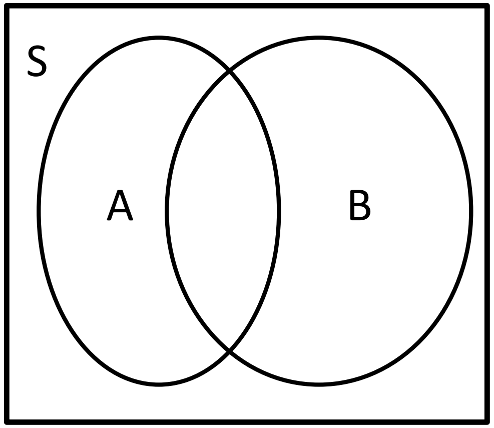
    $A \cup B = B \cup A$
  - Properties

    $A \cup A =$

    $A \cup S =$

    $A \cup \phi =$

    If $A \subset B$, $A \cup B =$



SET THEORY

- **Intersection (product)**
  - The intersection of two sets is the set consisting of all the elements from both sets.
  - Denoted as $A \cap B$
  - Associative law

    $$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$$
  - Commutative law

    $$A \cap B = B \cap A$$
  - Properties

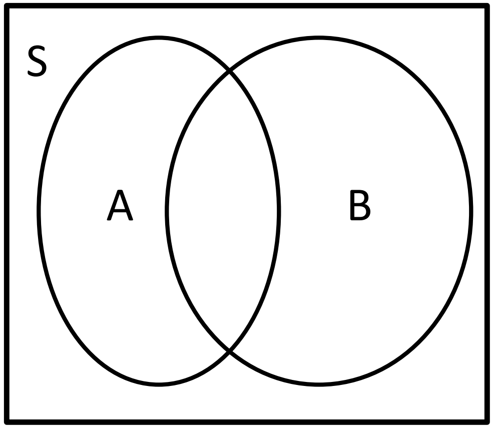
    $$A \cup A =$$

    $$A \cup S =$$

    $$A \cup \phi =$$

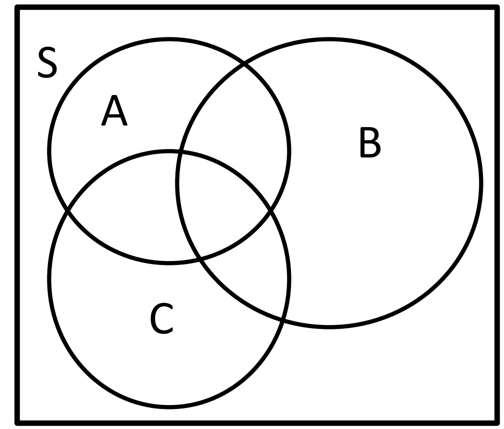
    if $A \subset B$, $A \cap B =$
  - Mutually exclusive (disjoint)

    A and B are mutually exclusive if $A \cap B = \phi$



SET THEORY

- **Intersection (Cont'd)**
  - $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$
  - $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  - $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$



- **Complement**
  - The complement of set A is a set containing all the elements of S that are not in A.
  - Denoted as A^c
  - Properties

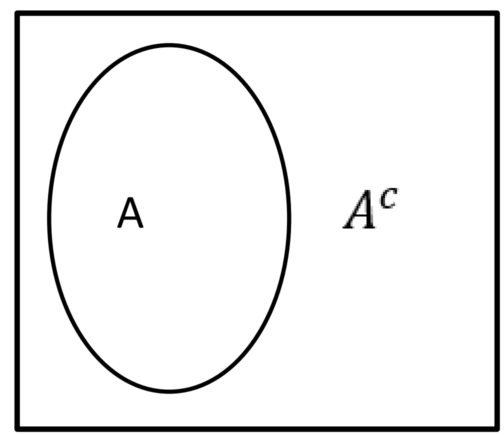
$\phi^c =$

$S^c =$

$(A^c)^c =$

$A \cup A^c =$

$A \cap A^c =$



SET THEORY

- **Complement (Cont'd)**
  - If $A \subset B$, then B^c A^c
  - DeMorgan's Laws

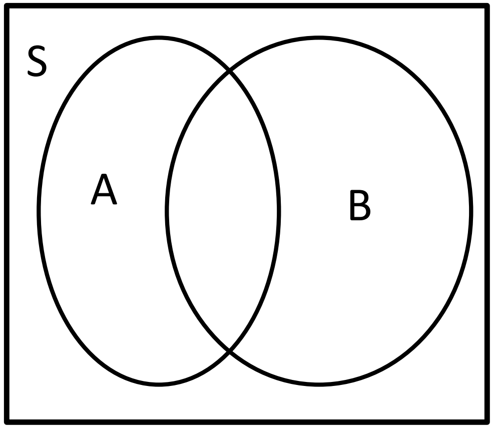
$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

SET THEORY

- **Difference**
  - The difference of two sets, *A*-*B*, is a set consisting of the elements of A that are not in B.

$$A - B = A \cap B^c$$



  - Examples

$$(A - B) \cup B =$$

$$(A - B) \cap B =$$

$$(A - B) \cap A =$$

$$A - \phi =$$

$$A - S =$$

$$S - A =$$

SET THEORY

- **Example**
  - S = {1, 2, 3, 4, 5, 6}
  - A = {2, 4, 6}, B = {1, 2, 3, 4}, C = {1, 3, 5}

$$(A \cup B) \cap C =$$

$$(A \cup B)^c =$$

$$(A \cap B)^c =$$

$$(A - B) \cap B =$$

$$(A \cup B) \cap (B - A) =$$

SET THEORY

- **Example**
  - If A and B are subset of the same space S
    - Simplify the following notation

$$(A \cap B) \cup (A - B) =$$

$$A^c \cap (A - B) =$$

$$(A \cap B) \cap (A \cup B) =$$

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RANDOM EXPERIMENT

- **Experiment:**
  - An *action* that results in an *outcome*.
- **Random experiment**
  - An experiment in which the outcome is *uncertain* before the experiment is performed.
- **Sample space S: the set of all possible outcomes of a random experiment**
  - Examples:
    - Flipping a coin
      - S = { H, T}
    - Flipping two coins
      - S = {(H, H), (H, T), (T, H), (T, T)}
    - Throwing a die
      - S = {1, 2, 3, 4, 5, 6}
    - Lifetime of a car
      - S = $[0, \infty)$
- **Outcome: each element in the sample space**

RANDOM EXPERIMENT

- **Random events: any subset *E* of the sample space *S***
  - E.g. throwing a die
    - E1 = {3}: the event that 3 appears
    - E2 = {2, 4, 6}: the event that an even number appears.
  - Elementary random event:
    - An elementary event is one for which there is only one element in the event
      - E.g. flipping a coin, {H} is an elementary event
      - E.g. tossing a dice, {4} is an elementary event
      - E.g. measuring a voltage, {2.37} Volt is an elementary event
  - Composite event
    - An event that might have several possible outcomes
      - E.g. tossing a die, "even" is a composite event, it includes the elementary outcomes {2, 4, 6}
      - E.g. measuring a voltage, "greater than 0" is a composite event, in includes all the elementary outcomes $(0, \infty)$

RANDOM EXPERIMENT

- **Trial**
  - A single performance of an experiment
  - Example: A coin is tossed 3 times, with outcomes H, H, T → 3 trials.
    - The 1st trial results in H, the 2nd trial results in H, the 3rd trial results in T
  - Example: Measure temperature twice, with outcomes 72.3 F, 71.8 F → 2 trials
  - Once a trial is performed, the result of the trial is deterministic (not random)

RANDOM EXPERIMENT

- **Classification: Discrete v.s. Continuous**
  - Discrete random experiment: the number of outcomes are countable (i.e., the outcomes can be put in a one-to-one correspondence with integers),
    - E.g. flipping a coin {H, T},
    - It is possible a discrete random experiment has infinite possible outcomes
      - E.g. count the number of atoms in an object: {1, 2, 3, …}
      - E.g. count the number of stars: {1, 2, 3, …}
  - Continuous random experiment: the number of outcomes are not countable
    - E.g. lifetime of a car $[0, \infty)$
    - E.g. randomly pick a real number in the range between [0, 1]
    - Continuous random experiment always has infinite outcomes

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PROBABILITY

- **Probability**
  - Assign a number between [0, 1] to each random event of a sample space, such that the number is a measure of how likely the event is.
    - E.g. flipping a coin. Assign 0.5 to H, 0.5 to T
      - 0.5 is the probability of H, 0.5 is the probability of T
    - E.g. tossing a die with events {1, 2} (2 or less)
      - 1/3 is the probability of "2 or less"
    - It is a mapping from a random event to a number between [0, 1]

Random event (a subset of S) → [0, 1]

  - There are several different definitions of probability
    - Relative-frequency approach
    - Axiomatic approach

PROBABILITY: RELATIVE-FREQUENCY

- **Relative-frequency**
  - Example: consider a random experiments with sample space: S ={A, B, C}. Perform the experiments for a total of N = 1,000 times
    - Event {A} occurs N_A = 500 times
    - Event {B} occurs N_B = 300 times
    - Event {C} occurs N_C = 200 times

    - $N_A + N_B + N_c = N$
    - The relative frequency of {A} is $r(A) = \frac{N_A}{N}$
    - The relative frequency of {B} is $r(B) = \frac{N_B}{N}$
    - The relative frequency of {C} is $r(C) = \frac{N_C}{N}$

    - The more trials we perform, the more accurate the result is.

PROBABILITY: RELATIVE-FREQUENCY DEFINITION

- **Relative-frequency**
  - The probability of event {A}

$$\Pr(A) = \lim_{N \to \infty} \frac{N_A}{N}$$

  - The sum of the probabilities of all the events

$$\Pr(A) + \Pr(B) + \Pr(C) =$$

PROBABILITY: RELATIVE FREQUENCY

- **Example**
  - Consider a bin contains 100 diodes. Among them, 20 are known to be bad.
  - Pick 1 diode from the bin
    - Sample space
    - Probability that the picked one is bad

PROBABILITY: AXIOMATIC DEFINITION

- **Definition: Probability**
  - Consider a random experiment with sample space S. For each event E of the sample space S, we assign it a positive number Pr(E), which satisfies the following properties
    - 1. $0 \le \Pr(E) \le 1$
    - 2. $\Pr(S) = 1$
    - 3. For any sequence of events $E_1, E_2, \cdots$ that are mutually exclusive $(E_m \cap E_n = \phi, \forall m \neq n)$

$$\Pr\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \Pr(E_n)$$

  - Then Pr(E) is called as the probability of event E.

PROBABILITY

- **Example**
  - Toss a die.
    - Sample space
    - The probability of getting a number 3
    - The probability of getting an even number

PROBABILITY

- **Probability of complement events**
  - Events E and E^c are always mutually exclusive

$$\Pr(E) + \Pr(E^c) =$$

- **Probability of two events**
  - Consider a sample space S with two events E and F. The probability that either E or F happens (the probability of all outcomes either in E or F)

$$\Pr(E \cup F) =$$

PROBABILITY

- **The probability of two events: consider a sample space *S* with two events *E* and *F***
  - The probability either *E* or *F* happens (the probability that the outcome is either in *E* or *F*)

$$\Pr(E \cup F)$$

  - The probability that both *E* and *F* happens (the probability that the outcome is in both *E* and *F*)

$$\Pr(E \cap F)$$

    - Also denoted as

$$\Pr(EF)$$

    - Called the joint probability

PROBABILITY

- **Property**
  - $$\Pr(A \cap B^c) + \Pr(A \cap B) = \Pr(A)$$

    - Proof:

      - Recall: $$(A \cap B^c) \cup (A \cap B) = A$$

        $$(A \cap B^c) \cap (A \cap B) = \emptyset$$

PROBABILITY

- **Property**

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

  – Proof:

PROBABILITY

- **Example**
  - Toss 2 different coins,
  - E = {(H, H), (H, T)}, F = {(H, H), (T, H)}, G={(T, T)}

$$\Pr(E \bigcup F) =$$

$$\Pr(E \cap F) =$$

$$\Pr(E \bigcup F \bigcup G) =$$

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CONDITIONAL PROBABILITY

- **Joint probability**
  - Consider a sample space S with two events E and F. The probability that both E and F happens (the probability of all outcomes in both E and F)
    - $\Pr(E \cap F)$
    - Also denoted as $\Pr(EF)$

CONDITIONAL PROBABILITY

- **Example**
  - Consider an urn contains 100 resistors of different resistance and power ratings. The number of the different types are listed as follows

	1 Ohm	10 Ohm	100 Ohm	Totals
1 W	10	20	30	60
10 W	10	20	10	40
Totals	20	40	40	100

  - Pick 1 resistor
  - What is the probability that the resistor has a power rating of 1 W?
  - What is the probability that the resistor has a resistance of 10 Ohm?
  - What is the probability that the resistor is 10 Ohm with 1 W rating?

CONDITIONAL PROBABILITY

- **Conditional Probability**
  - Pr(E|F): Given the condition that the event F occurred, the probability that E occurs.
  - Example:
    - (Cont'd from the previous example) Pick one resistor. If the picked resistor has a 1W rating, what is the probability that the resistor is 10 Ohm?

	1 Ohm	10 Ohm	100 Ohm	Totals
1 W	10	20	30	60
10 W	10	20	10	40
Totals	20	40	40	100

- Pr(10 Ohm|1W) =

- Pr(10Ohm, 1W) =Pr(10 Ohm|1W)Pr(1W)

CONDITIONAL PROBABILITY

- **Conditional Probability**

$$\Pr(A \mid B) = \frac{\Pr(AB)}{\Pr(B)}$$

$$\Pr(AB) = \Pr(A \mid B)\Pr(B) = \Pr(B \mid A)\Pr(A)$$

CONDITIONAL PROBABILITY

- **Example**
  - Consider a bin contains 100 diodes. Among them, 20 are known to be bad. Pick 2 diodes from the bin
    - Sample space
    - The probability that the 1st one is bad
    - If the 1st one is bad, what is the probability that the 2nd one is bad?
    - If the 1st one is good, what is the probability that the 2nd one is bad?
    - The probability that both are bad.
    - The probability that the 1st one is good, the 2nd one is bad.

CONDITIONAL PROBABILITY

- **Example**
  - Cards numbered 1 through 10 are placed in a hat. Pick one card. If we are told that the number on the card is at least 5. What is the probability that the card is 10?

CONDITIONAL PROBABILITY

- **Example**
  - Suppose an urn contains 7 black balls and 5 white balls. We draw 2 balls from the urn without replacement. What is the probability that both balls are black?

CONDITIONAL PROBABILITY

- **Example**
  - Alice can take either Chemistry or Physics. If Alice takes Chemistry, the probability that she will get an A is 0.5. If she takes Physics, the probability that she will get an A is 0.6. If Alice decides which course to take by flipping a fair coin. What is the probability that Alice gets an A in Chemistry?

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INDEPENDENCE

- **Independence**
  - Two events, A and B, are independent, if and only if

$$\Pr(AB) = \Pr(A)\Pr(B)$$

  - If A and B are independent

$$\Pr(A \mid B) =$$

$$\Pr(B \mid A) =$$

  - Independent $\neq$ mutually exclusive
    - A and B are independent

$$\Pr(A \cap B) = \Pr(A)\Pr(B) \qquad \Pr(A \cup B) =$$

    - A and B are mutually exclusive

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) \qquad \Pr(A \cap B) =$$

INDEPENDENCE

- **Example**
  - Toss a fair coin twice. Are the two trials independent?
  - Suppose an urn contains 7 black balls and 5 white balls. We draw 2 balls from the urn without replacement. Are the result of the 1^{st} draw independent of the 2^{nd} draw?

INDEPENDENCE

- **Example**
  - Toss 2 dice. Define event A as getting a 5, and event B as getting a 7.
    - Are A and B independent?
    - Are A and B mutually exclusive?

- **Example**
  - Toss 2 dice. Define event A as getting an odd number, and event B as getting a 7.
    - Are A and B independent?
    - Are A and B mutually exclusive?

INDEPENDENCE

- **Example**
  - A deck of 52 cards. Let A be the event of selecting an Ace, and let B be the event of selecting a Heart. 1. Are A and B independent? Are A and B mutually exclusive?
    - 4 suits: {club, spade, heart, diamond}
    - 13 denominations: {Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King}

INDEPENDENCE

- **Pairwise independence**
  - A sequence of events E_1, E_2, Λ, E_n are called pairwise independent if any pair of events are independent

$$\Pr(E_i E_j) = \Pr(E_i)\Pr(E_j) \qquad \forall i \neq j$$

- **Independence**
  - A sequence of events $E_1, E_2, \cdots, E_n$ are independent if any subset of the sequence are independent

$$\Pr(E_{r_1} E_{r_2} \cdots E_{r_k}) = \Pr(E_{r_1})\Pr(E_{r_2})\cdots\Pr(E_{r_k}) \qquad 1 \leq r_1, r_2, \cdots, r_k \leq n$$

  - Example

$$E_1, E_2, E_3, E_4$$

INDEPENDENCE

- **Example**
  - Let a ball be drawn from an urn containing 4 balls, numbered 1, 2, 3, 4. Let E = {1, 2}, F = {1, 3}, G = {1, 4}.
    - Are E, F, G pairwise independent?
    - Are E, F, G independent?

INDEPENDENCE: COMBINED EXPERIMENT

- **Combined experiment**
  - Example: a combined experiment is performed in which a coin is flipped and a single die is rolled.
    - Write the sample space.
    - Let A be the event of obtaining a head and a number 3 or less. Find the probability of A.

INDEPENDENCE: COMBINED EXPERIMENT

- **Combined experiment**
  - Two or more experiments are performed together, forming a combined experiment.
  - Experiment 1 has a sample space $S_1 = \{a_1, a_2, \cdots, a_n\}$
  - Experiment 2 has a sample space $S_2 = \{b_1, b_2, \cdots, b_m\}$
  - Then the sample space of the combined experiment is

$$S = S_1 \times S_2 = \{(a_1, b_1), (a_1, b_2), \cdots, (a_1, b_m), \cdots, (a_n, b_1), \cdots, (a_n, b_m)\}$$

  - $S = S_1 \times S_2$ is called the Cartesian product of S_1 and S_2

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BAYES' FORMULA

- **Total probability**
  - If we know P(A|B), how do we find out P(A)
    - Conditional probability ➔ Unconditional probability
  - Suppose $B_1, B_2, \cdots, B_n$ are mutually exclusive events and $B_1 \bigcup B_2 \bigcup \cdots \bigcup B_n = S$

$$\Pr(A) = \Pr(A \mid B_1)\Pr(B_1) + \Pr(A \mid B_2)\Pr(B_2) + \cdots + \Pr(A \mid B_n)\Pr(B_n)$$

    - Proof

BAYES' FORMULA

- **Example**
  - Consider two urns. The 1st contains 3 white and 7 black balls. The 2nd contains 8 white and 2 black balls. We flip a fair coin and then draw a ball from the 1st urn if the coin is head, and draw a ball from the 2nd urn if the coin is tail.
    - What is the probability drawing from the 1st urn?
    - What is the probability of drawing a black ball from the 1st urn?
    - What is the probability of drawing a black ball from the 2nd urn?
    - What is the probability of drawing a black ball?

BAYES' FORMULA

- **Bayes' formula**
  - $\Pr(A|B_i) \rightarrow \Pr(B_i|A)$
  - Suppose $B_1, B_2, \ldots, B_n$ are mutually exclusive events and $B_1 \cup B_2 \cup \cdots \cup B_n = S$

$$\Pr(B_i \mid A) = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\Pr(A)} = \frac{\Pr(A \mid B_i)\Pr(B_i)}{\sum_{k=1}^{n} \Pr(A \mid B_k)\Pr(B_k)}$$

  - Proof

BAYES' FORMULA

- **Example**
  - Consider two urns. The 1st contains 3 white and 7 black balls. The 2nd contains 8 white and 2 black balls. We flip a fair coin and then draw a ball from the 1st urn if the coin is head, and draw a ball from the 2nd urn if the coin is tail.
    - If we draw a black ball, what is the probability that we draw it from the 1st urn?
    - If we draw a black ball, what is the probability that we draw it from the 2nd urn?

BAYES' FORMULA

- **Example**
  - Alice can take either Chemistry or Physics. If Alice takes Chemistry, the probability that she will get an A is 0.5. If she takes Physics, the probability that she will get an A is 0.6. If Alice decides which course to take by flipping a fair coin. If Alice gets an A, what is the probability that she takes Chemistry?

BAYES' FORMULA

- **Example**
  - A manufacture buys components in equal amount from 4 different suppliers. The probability that components from supplier 1 are bad is 0.05, that components from supplier 2 are bad is 0.1, that components from supplier 3 are bad is 0.2, that components from supplier 4 are bad is 0.15,
    - The probability that a component is bad.
    - If a component is bad, the probability that it is from supplier 2.

BAYES' FORMULA

- **Example**
  - A lab blood test to detect a certain disease. If the disease is present, the test can detect it 95% of the time. However, the test also gives a "false positive" result for 1% of the healthy person being tested (that is, if a healthy person is tested, then, with probability 0.01, the result will imply the person has the disease). Assume 0.5% of the population actually has the disease. If the test result of a person is positive, what is the probability that the person actually has the disease?

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BERNOULLI TRIALS

- **Bernoulli trials**
  - Consider an experiment with event *A*, and Pr(*A*) = *p*.
  - Repeat the experiment *n* times, what is the probability that event A happens exactly *k* times? $(0 \le k \le n)$
  - The composite events that there are exactly *k* event *A* in *n* trials:
  - The probability of one of the events
  - How many such events are there?
    - Choose *k* position out of *n* positions (example, k = 2, n = 4)

BERNOULLI TRIALS

- **Bernoulli trials**

$$p_n(k) = \Pr(\text{A occurs k times in n trials}) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

BERNOULLI TRIALS

- **Example**
  - Toss an unfair coin 6 times. The probability of Head is 0.3.
    - What is the probability that the head occurs twice?
    - What is the probability that the head occurs more than once?