Department of Electrical Engineering
University of Arkansas

ELEG 3124 SYSTEMS AND SIGNALS
Ch. 5 Fourier Transform

Dr. Jingxian Wu
wuj@uark.edu

OUTLINE

- **Introduction**
- **Fourier Transform**
- **Properties of Fourier Transform**
- **Applications of Fourier Transform**

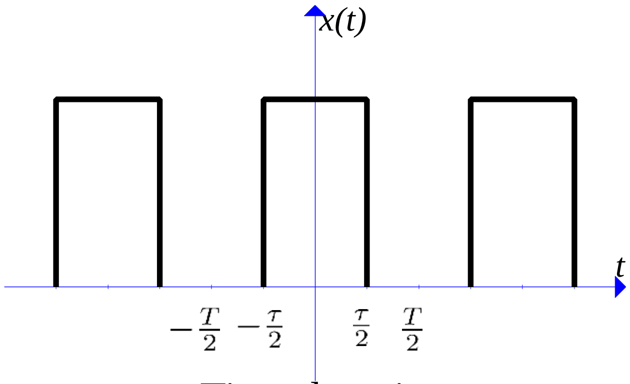
INTRODUCTION: MOTIVATION

- **Motivation:**
  - Fourier series: periodic signals can be decomposed as the summation of orthogonal complex exponential signals

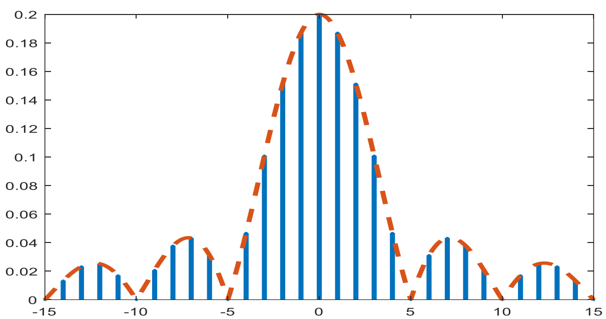
$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \exp[jn\omega_0 t]$$

$$c_n = \frac{1}{T}\int_0^T x(t)\exp[jn\omega_0 t]dt$$

- each harmonic contains a unique frequency: *n/T*



Time domain



$\tau = 1, T = 5$

Frequency domain

- time domain ←→ frequency domain

How about aperiodic signals $(T = \infty)$ **?**

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INTRODUCTION: TRANSFER FUNCTION

- **System transfer function**

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow e^{j\omega t}H(\omega)$$

$$H(\omega) = \int_{-\infty}^{+\infty} h(t)\exp\left[j\omega t\right]dt$$

- **System with periodic inputs**

$$e^{jn\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow e^{jn\omega_0 t}H(n\omega_0)$$

$$\sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} H(n\omega_0)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} H(n\omega_0)$$

OUTLINE

- Introduction
- **Fourier Transform**
- Properties of Fourier Transform
- Applications of Fourier Transform

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FOURIER TRANSFORM

- **Fourier Transform**

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

  – given $x(t)$, we can find its Fourier transform $X(\omega)$

- **Inverse Fourier Transform**

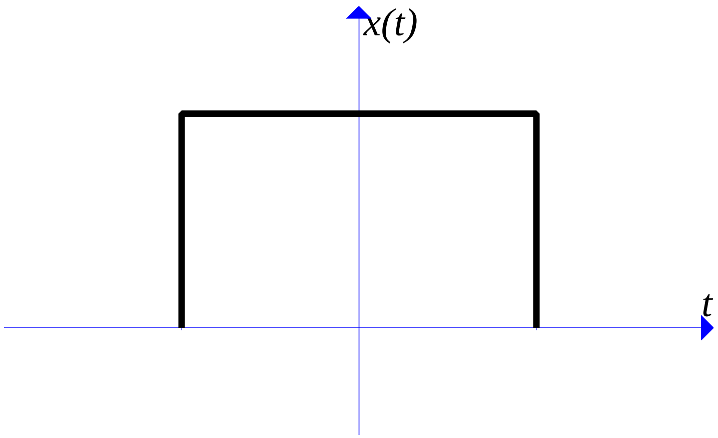
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

  – given $X(\omega)$, we can find the time domain signal $x(t)$
  – signal is decomposed into the "weighted summation" of complex exponential functions. (integration is the extreme case of summation)

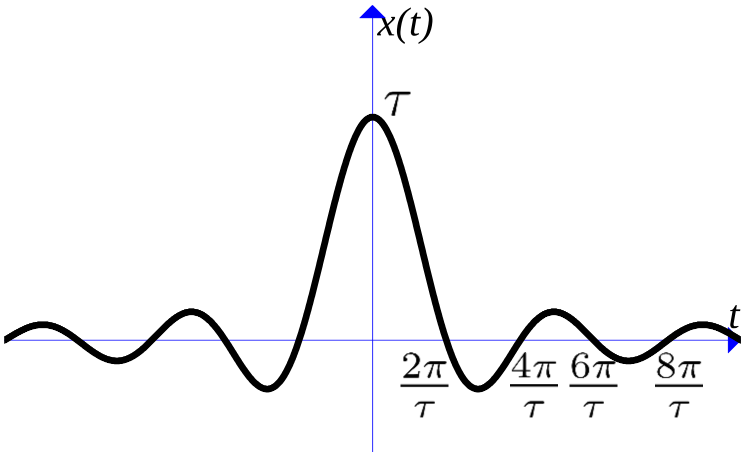
$$x(t) \Leftrightarrow X(\omega)$$

FOURIER TRANSFORM

- **Example**
  - Find the Fourier transform of $x(t) = rect(t/\tau)$



$$x(t) = \text{rect}\left(\frac{t}{\tau}\right)$$



$$X(\omega) = \tau \text{sinc} \frac{\omega\tau}{2\pi}$$

FOURIER TRANSFORM

- **Example**
  - Find the Fourier transform of $x(t) = \exp(-a\,|\,t\,|)$ $\quad a > 0$

FOURIER TRANSFORM

- **Example**
  - Find the Fourier transform of $x(t) = \exp(-at)u(t)$ $\quad a > 0$

FOURIER TRANSFORM

- **Example**
  - Find the Fourier transform of $x(t) = \delta(t - a)$

FOURIER TRANSFORM: TABLE

$x(t)$	$X(\omega)$
1	$2\pi\delta(\omega)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-t_0)$	$\exp(-j\omega t_0)$
$\exp(j\omega_0 t)$	$2\pi\delta(\omega-\omega_0)$
$\mathrm{rect}(t/\tau)$	$\tau \mathrm{sinc} \frac{\omega\tau}{2\pi}$
$\mathrm{sinc}(t)$	$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$
$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
$\cos(\omega_0 t)$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin(\omega_0 t)$	$\frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$

$x(t)$	$X(\omega)$
$\exp(-at)u(t), \quad \Re(a)>0$	$\frac{1}{a+j\omega}$
$t\exp(-at)u(t), \quad \Re(a)>0$	$\frac{1}{(a+j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!}\exp(-at)u(t), \quad \Re(a)>0$	$\frac{1}{(a+j\omega)^n}$
$\exp(-a\|t\|), \quad a>0$	$\frac{2a}{a^2+\omega^2}$
$\|t\|\exp(-a\|t\|), \quad \Re(a)>0$	$\frac{4aj\omega}{a^2+\omega^2}$

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FOURIER TRANSFORM

- **The existence of Fourier transform**
  - Not all signals have Fourier transform
  - If a signal have Fourier transform, it must satisfy the following two conditions
    - 1. $x(t)$ is absolutely integrable

$$\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$$

    - 2. $x(t)$ is well behaved
      - The signal has finite number of discontinuities, minima, and maxima within any finite interval of time.
- **Example**
  - $x(t) = \exp(t)u(t)$

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OUTLINE

- Introduction
- Fourier Transform
- **Properties of Fourier Transform**
- Applications of Fourier Transform

PROPERTIES: LINEARITY

- **Linearity**
  - If $x_1(t) \Leftrightarrow X_1(\omega)$ $x_2(t) \Leftrightarrow X_2(\omega)$
  - then $ax_1(t) + bx_2(t) \Leftrightarrow aX_1(\omega) + bX_2(\omega)$

- **Example**
  - Find the Fourier transform of $x(t) = 2rect(t/\tau) + 3\exp(-2t)u(t) + 4\delta(t)$

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PROPERTY: TIME-SHIFT

- **Time shift**
  - If $x(t) \Leftrightarrow X(\omega)$
  - Then $x(t-t_0) \Leftrightarrow X(\omega)\exp[-j\omega t_0]$

phase shift

- **Review: complex number**

$$c = |c|e^{j\theta} = |c|\cos(\theta) + j|c|\sin(\theta) = a + jb$$

$$a = |c|\cos\theta \qquad b = |c|\sin\theta$$

$$|c| = \sqrt{a^2 + b^2} \qquad \theta = a\tan(b/a)$$

  - Phase shift of a complex number c by θ_0 : $c\exp(j\theta_0) = |c|\exp[j(\theta + \theta_0)]$

time shift in time domain ➔ frequency shift in frequency domain

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PROPERTY: TIME SHIFT

- **Example:**
  - Find the Fourier transform of $x(t) = rect[t-2]$

PROPERTY: TIME SCALING

- **Time scaling**
  - If $x(t) \Leftrightarrow X(\omega)$
  - Then $x(at) \Leftrightarrow \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

- **Example**
  - Let $X(\omega) = rect\left[(\omega - 1)/2\right]$, find the Fourier transform of $x(-2t+4)$

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PROPERTY: SYMMETRY

- **Symmetry**
  - If $x(t) \Leftrightarrow X(\omega)$, and $x(t)$ is a real-valued time signal
  - Then $X(-\omega) = X^*(\omega)$

PROPERTY: DIFFERENTIATION

- **Differentiation**
  - If $x(t) \Leftrightarrow X(\omega)$
  - Then $\dfrac{dx(t)}{dt} \Leftrightarrow j\omega X(\omega)$ $\qquad \dfrac{d^n x(t)}{dt^n} \Leftrightarrow (j\omega)^n X(\omega)$

- **Example**
  - Let $X(\omega) = rect[(\omega - 1)/2]$, find the Fourier transform of $\dfrac{dx(t)}{dt}$

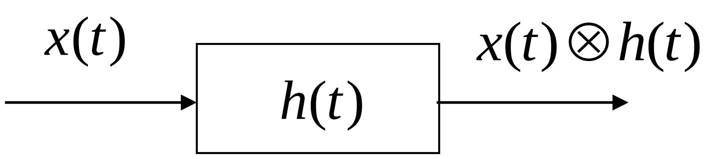
PROPERTY: DIFFERENTIATION

- **Example**
  - Find the Fourier transform of $x(t) = \mathrm{sgn}(t)$

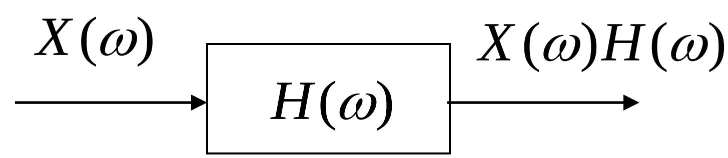
    (Hint: $\frac{d}{dt}\left[\frac{1}{2}\mathrm{sgn}(t)\right] = \delta(t)$)

PROPERTY: CONVOLUTION

- **Convolution**
  - If $x(t) \Leftrightarrow X(\omega)$, $h(t) \Leftrightarrow H(\omega)$
  - Then $x(t) \otimes h(t) \Leftrightarrow X(\omega)H(\omega)$



time domain



frequency domain

PROPERTY: CONVOLUTION

- **Example**
  - An LTI system has impulse response $h(t) = \exp(-at)u(t)$

    If the input is $x(t) = (a-b)\exp(-bt)u(t) + (c-a)\exp(-ct)u(t)$

    Find the output $(a > 0, b > 0, c > 0)$

PROPERTY: MULTIPLICATION

- **Multiplication**
  - If $x(t) \Leftrightarrow X(\omega)$, $m(t) \Leftrightarrow M(\omega)$
  - Then $$x(t)m(t) \Leftrightarrow \frac{1}{2\pi}\left[X(\omega) \otimes M(\omega)\right]$$

PROPERTY: DUALITY

- **Duality**
  - If $g(t) \Leftrightarrow G(\omega)$
  - Then $G(t) \Leftrightarrow 2\pi g(-\omega)$

PROPERTY: DUALITY

- **Example**
  - Find the Fourier transform of $h(t) = Sa\left(\frac{t}{2}\right)$

(recall: $\mathrm{rect}(t/\tau) \Leftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$)

PROPERTY: DUALITY

- **Example**
  - Find the Fourier transform of $x(t) = 1$

  - Find the Fourier transform of $x(t) = e^{j\omega_0 t}$

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PROPERTY: SUMMARY

Properties	time-domain	frequency-domain
Linearity	$\sum_{n=1}^{N} \alpha_n x_n(t)$	$\sum_{n=1}^{N} \alpha_n X_n(\omega)$
Time shift	$x(t-t_0)$	$X(\omega)\exp(-j\omega t_0)$
Frequency shift	$\exp(j\omega_0 t)x(t)$	$X(\omega-\omega_0)$
Time scaling	$x(\alpha t)$	$X(\omega/\alpha)/\lvert\alpha\rvert$
Differentiation	$d^n x(t)/dt^n$	$(j\omega)^n X(\omega)$
Multiplication by t	$(-jt)^n x(t)$	$\frac{d^n X(\omega)}{d\omega^n}$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
Convolution	$x(t) \otimes h(t)$	$X(\omega)H(\omega)$
Multiplication	$x(t)m(t)$	$\frac{1}{2\pi} X(\omega) \otimes M(\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$

PROPERTY: EXAMPLES

- **Examples**
  - 1. Find the Fourier transform of $x(t) = \cos(\omega_0 t)$

  - 2. Find the Fourier transform of $x(t) = u(t)$

$$u(t) = \frac{1}{2}\left[\operatorname{sgn}(t) + 1\right] \qquad \operatorname{sgn}(t) \leftrightarrow \frac{2}{j\omega}$$

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PROPERTY: EXAMPLES

- **Examples**
  - 3. A LTI system with impulse response $h(t) = \exp[-at]u(t)$

    Find the output when input is $x(t) = u(t)$

  - 4. If $x(t) \Leftrightarrow X(\omega)$, find the Fourier transform of $\int_{-\infty}^{t} x(\tau)d\tau$

    (Hint: $\int_{-\infty}^{t} x(\tau)d\tau = x(t) \otimes u(t)$)

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PROPERTY: EXAMPLES

- **Example**
  - 5. (Modulation) If $x(t) \Leftrightarrow X(\omega)$, $m(t) = \cos(\omega_0 t)$

    Find the Fourier transform of $x(t)m(t)$

  - 6. If $X(\omega) = \dfrac{1}{a + j\omega}$, find $x(t)$

PROPERTY: DIFFERENTIATION IN FREQ. DOMAIN

- **Differentiation in frequency domain**
  - If: $x(t) \Leftrightarrow X(\omega)$
  - Then: $(-jt)^n x(t) = \dfrac{d^n X(\omega)}{d\omega^n}$

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PROPERTY: DIFFERENTIATION IN FREQ. DOMAIN

- **Example**
  - Find the Fourier transform of $t\exp(-at)u(t), \quad a > 0$

PROPERTY: FREQUENCY SHIFT

- **Frequency shift**
  - If: $x(t) \Leftrightarrow X(\omega)$
  - Then: $x(t)\exp(j\omega_0 t) \Leftrightarrow X(\omega-\omega_0)$

- **Example**
  - If $X(\omega) = rect[(\omega-1)/2]$, find the Fourier transform $x(t)\exp(-j2t)$

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PROPERTY: PARSAVAL'S THEOREM

- **Review: signal energy**

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

- **Parsaval's theorem**

$$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega}$$

PROPERTY: PARSAVAL'S THEOREM

- **Example:**
  - Find the energy of the signal $x(t) = \exp(-2t)u(t)$

PROPERTY: PERIODIC SIGNAL

- **Fourier transform of periodic signal**
  - Periodic signal can be written as Fourier series

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \exp\left[jn\omega_0 t\right]$$

  - Perform Fourier transform on both sides

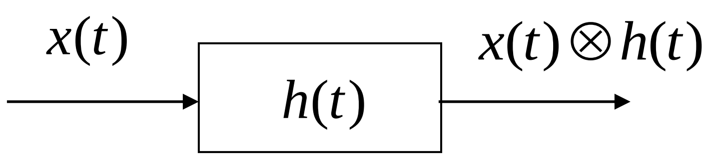
$$X(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} c_n \delta(\omega - n\omega_0)$$

OUTLINE

- Introduction
- Fourier Transform
- Properties of Fourier Transform
- **Applications of Fourier Transform**

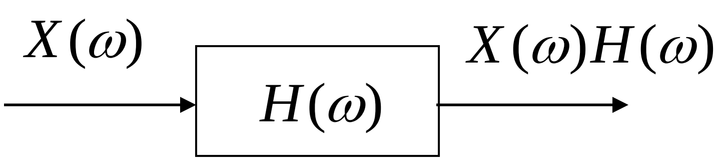
APPLICATIONS: FILTERING

- **Filtering**
  - Filtering is the process by which the essential and useful part of a signal is separated from undesirable components.
    - Passing a signal through a filter (system).
    - At the output of the filter, some undesired part of the signal (e.g. noise) is removed.
  - Based on the convolution property, we can design filter that only allow signal within a certain frequency range to pass through.



filter

time domain

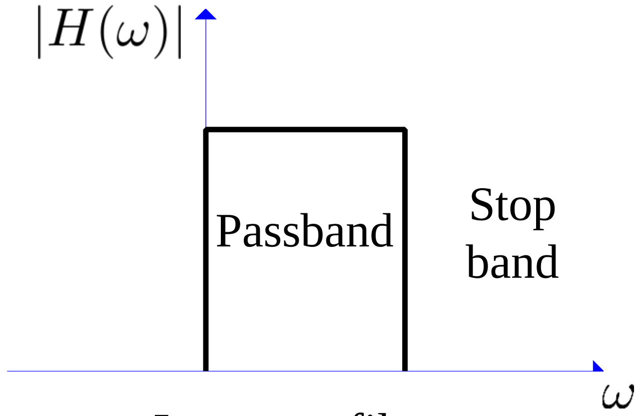


filter

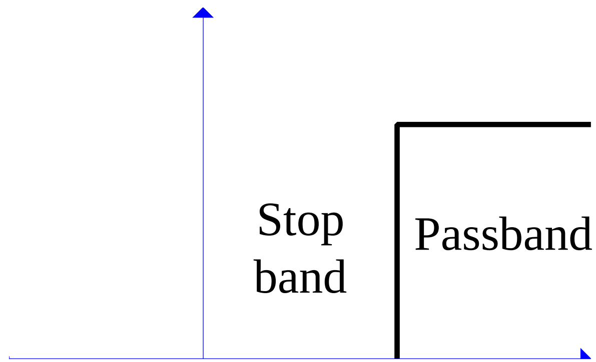
frequency domain

APPLICATIONS: FILTERING

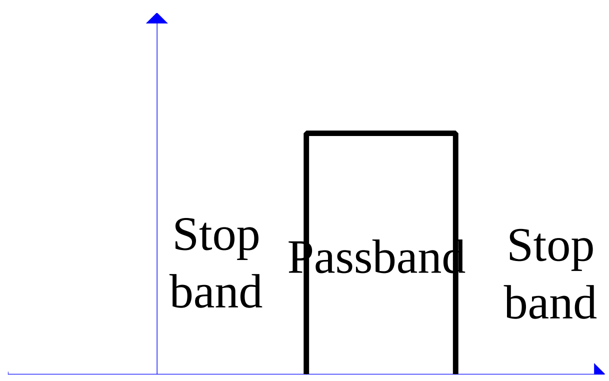
- **Classifications of filters**



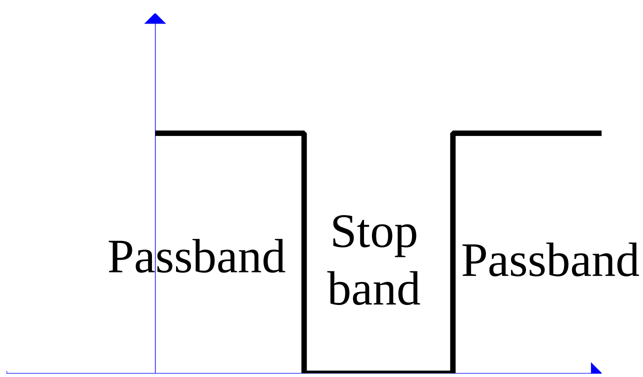
Low pass filter



High pass filter



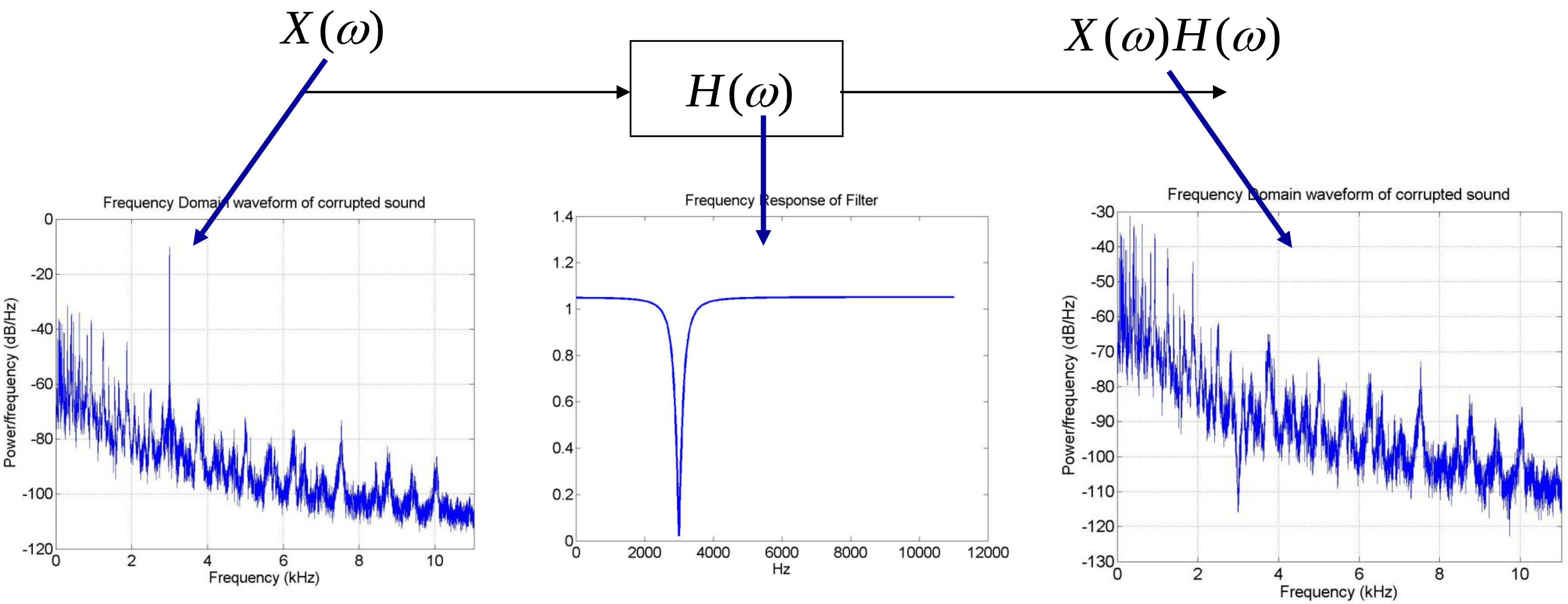
Band pass filter



Band stop (Notch) filter

APPLICATION: FILTERING

- **A filtering example**
  - A demo of a notch filter



Corrupted sound

Filter

Filtered sound

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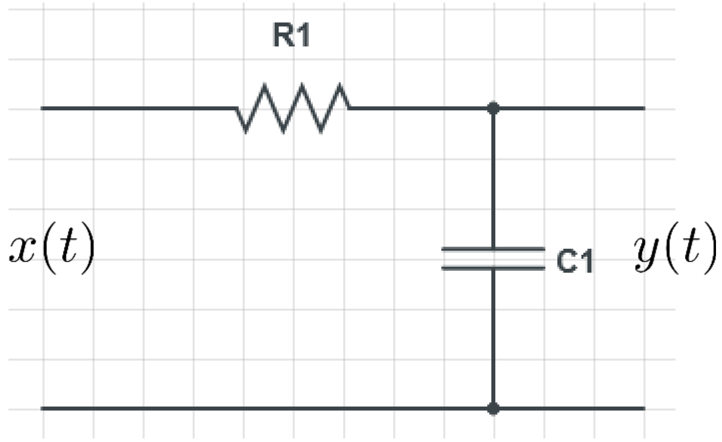
APPLICATIONS: FILTERING

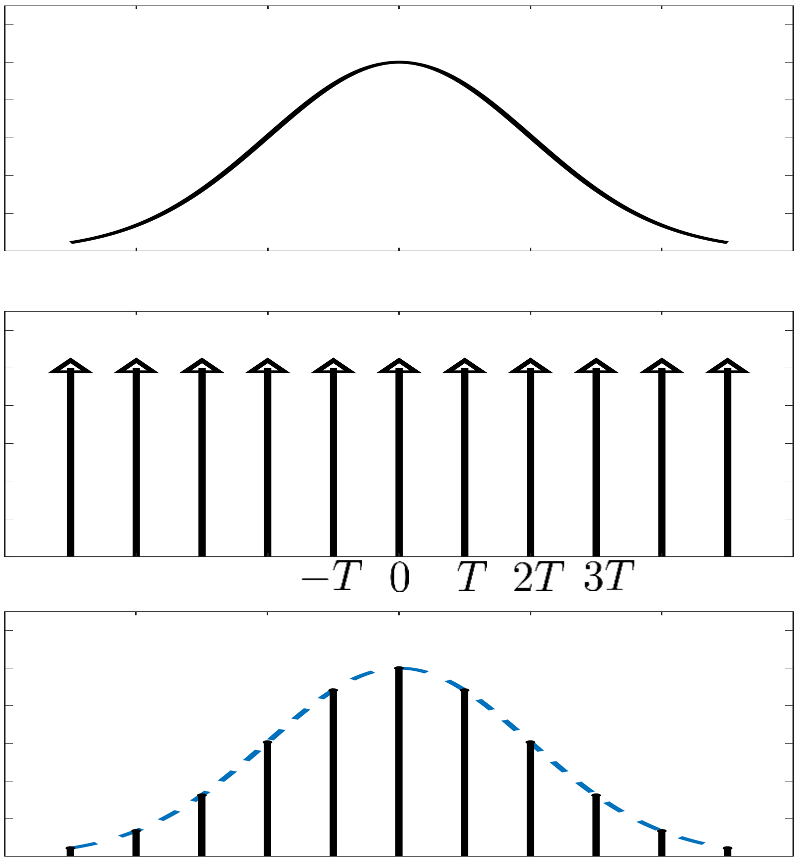
- **Example**
  - Find out the frequency response of the RC circuit
  - What kind of filters it is?



RC circuit

APPLICATION: SAMPLING THEOREM

- **Sampling theorem: time domain**
  - Sampling: convert the continuous-time signal to discrete-time signal.



Sampled signal

$$x(t)$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

sampling period

$$x_s(t) = x(t)\,p(t)$$

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APPLICATION: SAMPLING THEOREM

- **Sampling theorem: frequency domain**
  - Fourier transform of the impulse train
    - impulse train is periodic

Fourier series

$$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT_s)=\frac{1}{T_s}\sum_{n=-\infty}^{+\infty}1\times e^{jn\omega_s t}$$

$$\omega_s=\frac{2\pi}{T_s}$$

- Find Fourier transform on both sides

$$P(\omega)=\frac{2\pi}{T_s}\sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_s)$$

- Time domain multiplication ➔ Frequency domain convolution

$$x(t)p(t)\Leftrightarrow\frac{1}{2\pi}\left[X(\omega)\otimes P(\omega)\right]$$

$$x(t)p(t)\Leftrightarrow\frac{1}{T_s}\sum_{n=-\infty}^{+\infty}X(\omega-n\omega_s)$$

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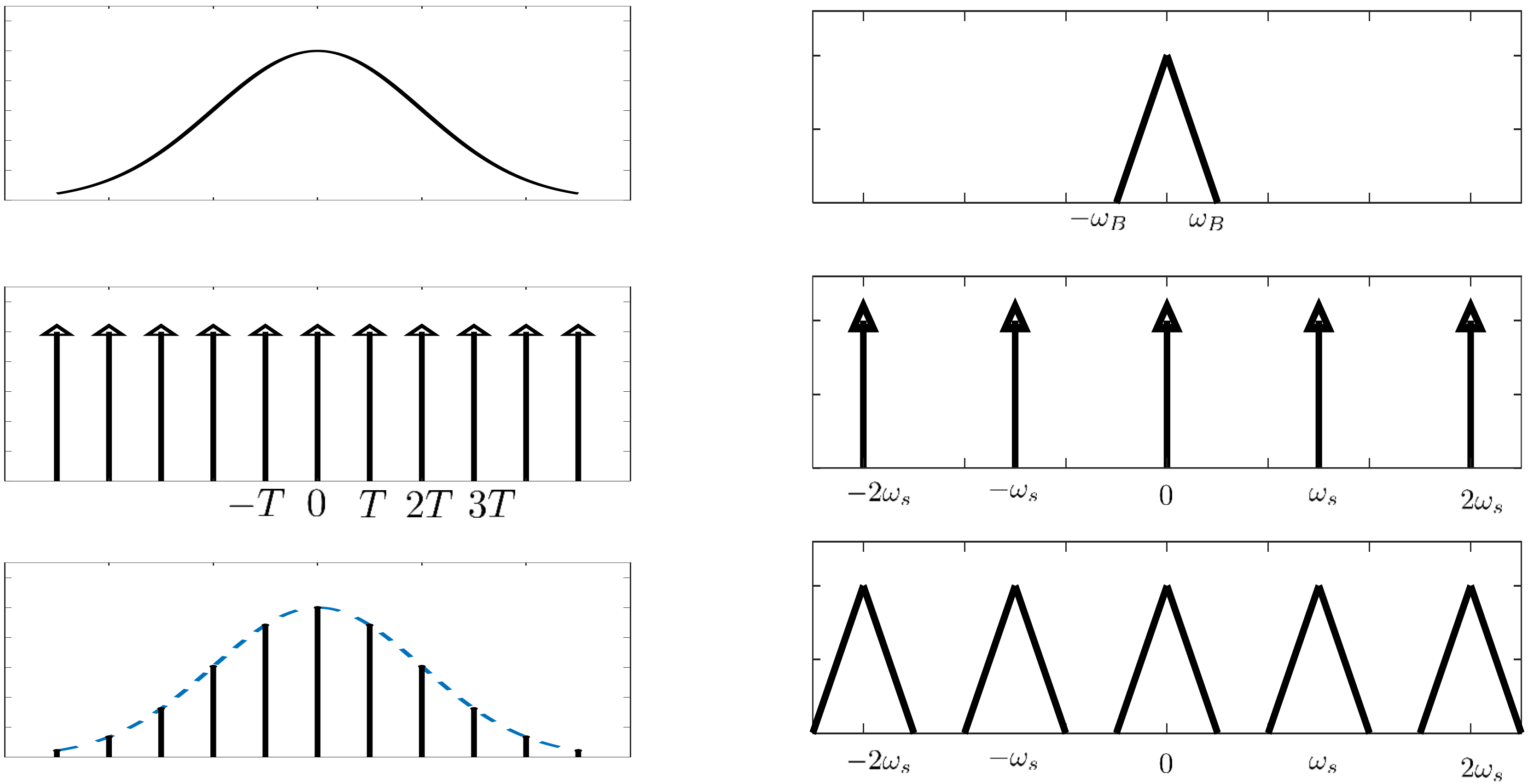
APPLICATION: SAMPLING THEOREM

- **Sampling theorem: frequency domain**
  - Sampling in time domain ➔ Repetition in frequency domain



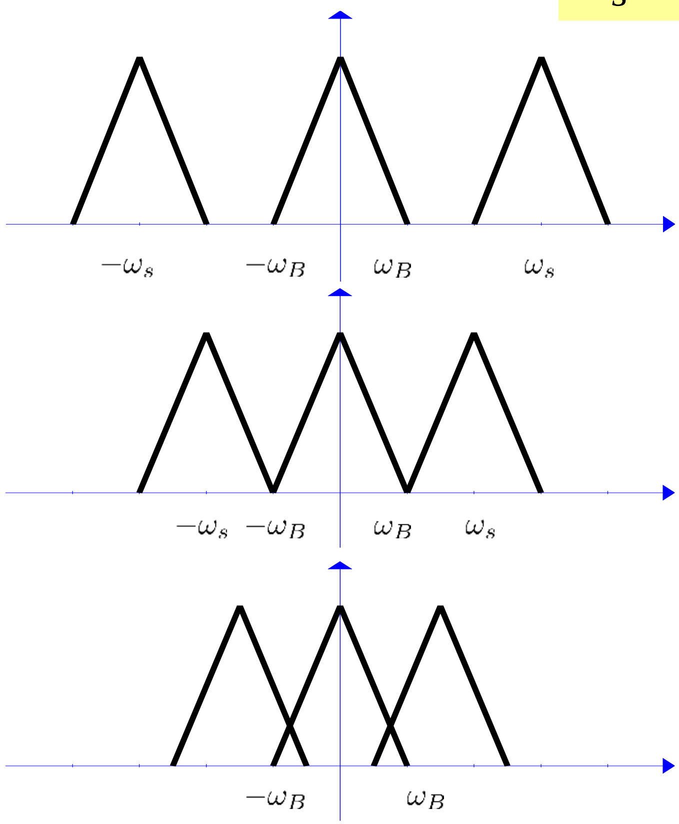
Time domain

Frequency domain

APPLICATION: SAMPLING THEOREM

- **Sampling theorem**
  - If the sampling rate is twice of the bandwidth, then the original signal can be perfectly reconstructed from the samples.

$$\omega_s > 2\omega_B$$



$$\omega_s > 2\omega_B$$

$$\omega_s = 2\omega_B$$

$$\omega_s < 2\omega_B$$

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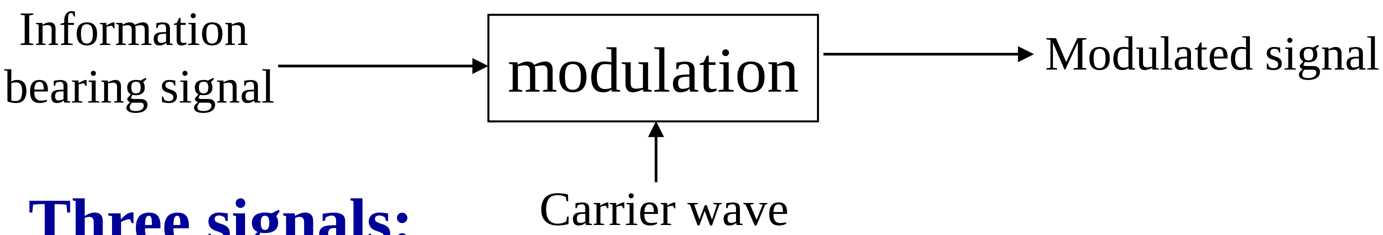
Frequency domain

APPLICATION: AMPLITUDE MODULATION

- **What is modulation?**
  - The process by which some characteristic of a carrier wave is varied in accordance with an information-bearing signal



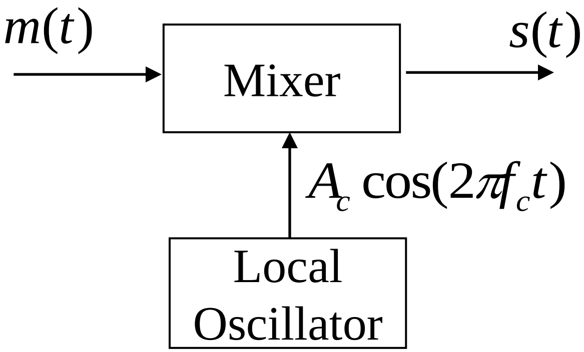
- **Three signals:**
  - Information bearing signal (modulating signal)
    - Usually at low frequency (baseband)
    - E.g. speech signal: 20Hz – 20KHz
  - Carrier wave
    - Usually a high frequency sinusoidal (passband)
    - E.g. AM radio station (1050KHz) FM radio station (100.1MHz), 2.4GHz, etc.
  - Modulated signal: passband signal

APPLICATION: AMPLITUDE MODULATION

- **Amplitude Modulation (AM)**

$$s(t) = A_c m(t)\cos(2\pi f_c t)$$

– A direct product between message signal and carrier signal

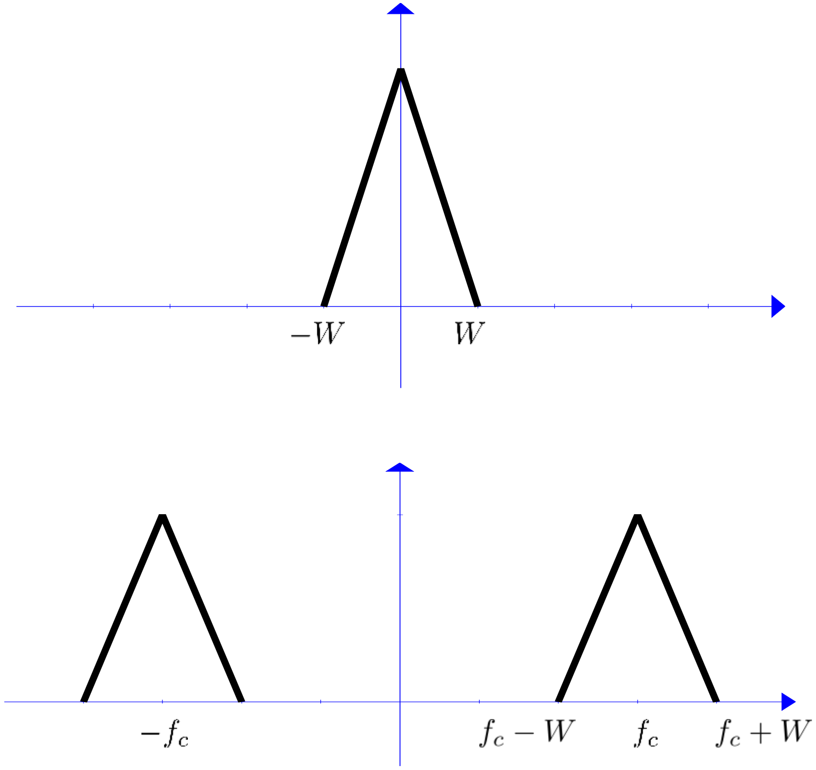


Amplitude modulation

APPLICATION: AMPLITUDE MODULATION

- **Amplitude Modulation (AM)**

$$S(f)=\frac{A_c}{2}\left[M(f-f_c)+M(f+f_c)\right]$$



Amplitude modulation