Department of Electrical Engineering
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ELEG 3124 SYSTEMS AND SIGNALS
Ch. 4 Fourier Series

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OUTLINE

- **Introduction**
- **Fourier series**
- **Properties of Fourier series**
- **Systems with periodic inputs**

INTRODUCTION: MOTIVATION

- **Motivation of Fourier series**
  - Convolution is derived by decomposing the signal into the sum of a series of delta functions
    - Each delta function has its unique delay in time domain.
    - Time domain decomposition

$$x(t)=\int_{-\infty}^{+\infty}x(\tau)\delta(t-\tau)d\tau=\lim_{\Delta\to 0}\sum_{n=-\infty}^{+\infty}x(n\Delta)\delta(t-n\Delta)\Delta$$

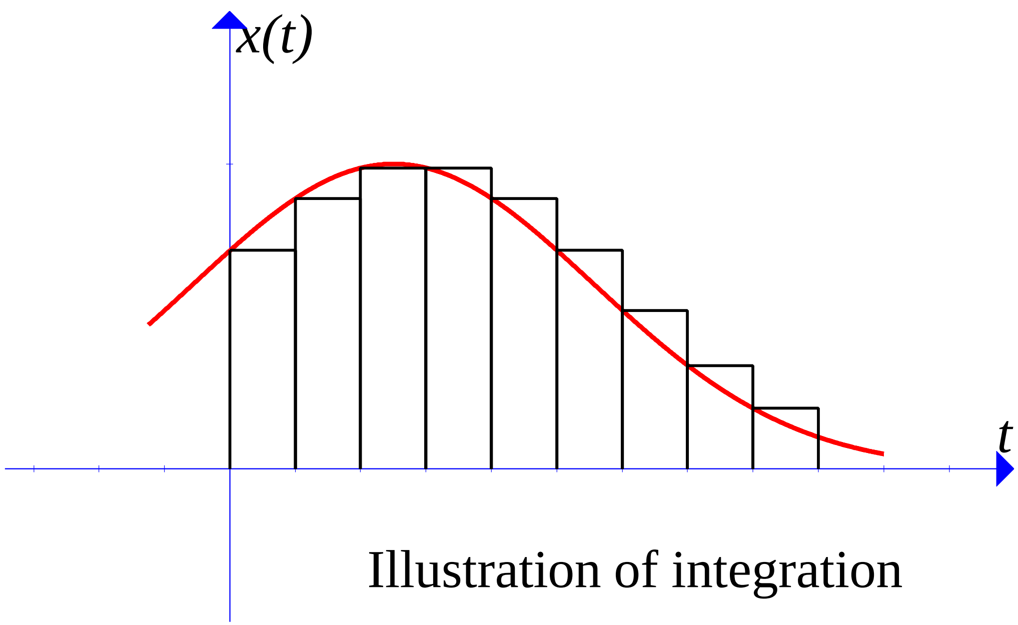


Illustration of integration

INTRODUCTION: MOTIVATION

- **Can we decompose the signal into the sum of other functions**
  - Such that the calculation can be simplified?
  - Yes. We can decompose periodic signal as the sum of a sequence of complex exponential signals ➔ Fourier series.

$$e^{j\Omega_0 t} = e^{j2\pi f_0 t}$$

$$f_0 = \frac{\Omega_0}{2\pi}$$

  - Why complex exponential signal? (what makes complex exponential signal so special?)
    - 1. Each complex exponential signal has a unique frequency ➔ frequency decomposition
    - 2. Complex exponential signals are periodic

INTRODUCTION: REVIEW

- **Complex exponential signal**

$$e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$$

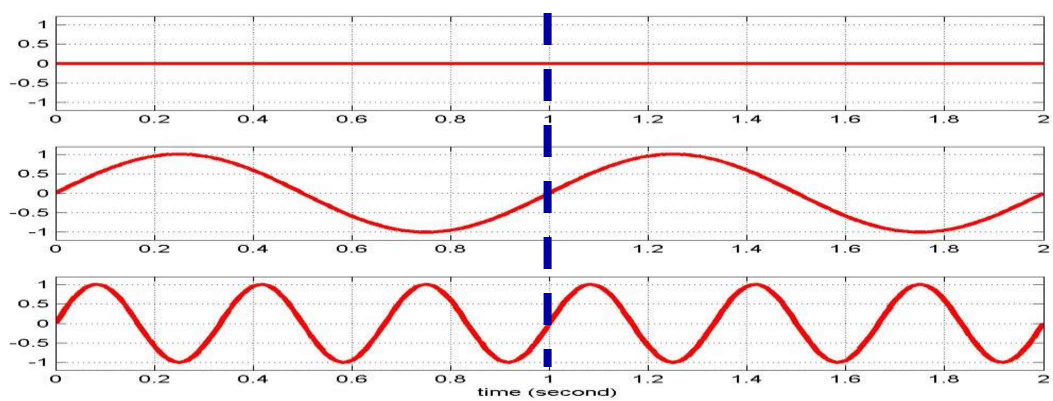
  – Complex exponential function has a one-to-one relationship with sinusoidal functions.

  – Each sinusoidal function has a unique frequency: f

- **What is frequency?**

  – Frequency is a measure of how fast the signal can change within a unit time.

    - Higher frequency ➔ signal changes faster



f = 0 Hz

f = 1 Hz

f = 3 Hz

Sinusoidal at different frequencies

INTRODUCTION: ORTHONORMAL SIGNAL SET

- **Definition: orthogonal signal set**
  - A set of signals, $\{\phi_0(t), \phi_1(t), \phi_2(t), \cdots\}$, are said to be orthogonal over an interval (a, b) if

$$\int_a^b \phi_l(t)\phi_k^*(t)dt = \begin{cases} C, & l = k \\ 0, & l \neq k \end{cases}$$

- **Example:**
  - the signal set: $\phi_k(t) = e^{jk\Omega_0 t}$ $k = 0, \pm 1, \pm 2, \cdots$ are orthogonal over the interval $[0, T_0]$, where $\Omega_0 = \dfrac{2\pi}{T_0}$

OUTLINE

- Introduction
- **Fourier series**
- Properties of Fourier series
- Systems with periodic inputs

FOURIER SERIES

- **Definition:**
  - For any periodic signal with fundamental period T_0, it can be decomposed as the sum of a set of complex exponential signals as

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

$$\Omega_0 = \frac{2\pi}{T_0}$$

- $c_n, n = 0, \pm 1, \pm 2, \cdots$, Fourier series coefficients

$$c_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-jn\Omega_0 t} dt$$

- derivation of c_n :

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FOURIER SERIES

- **Fourier series**

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

  - The periodic signal is decomposed into the weighted summation of a set of orthogonal complex exponential functions.
  - The frequency of the n-th complex exponential function: $n\Omega_0$
    - The periods of the n-th complex exponential function: $T_n = \frac{T_0}{n}$
  - The values of coefficients, $c_n, n = 0, \pm1, \pm2, \cdots$, depend on $x(t)$
    - Different x(t) will result in different c_n
    - There is a one-to-one relationship between $x(t)$ and c_n

$$s(t) \Longleftrightarrow [\cdots, c_{-2}, c_{-1}, c_0, c_1, c_2, \cdots]$$

For a periodic signal, it can be either represented as $s(t)$, or represented as c_n

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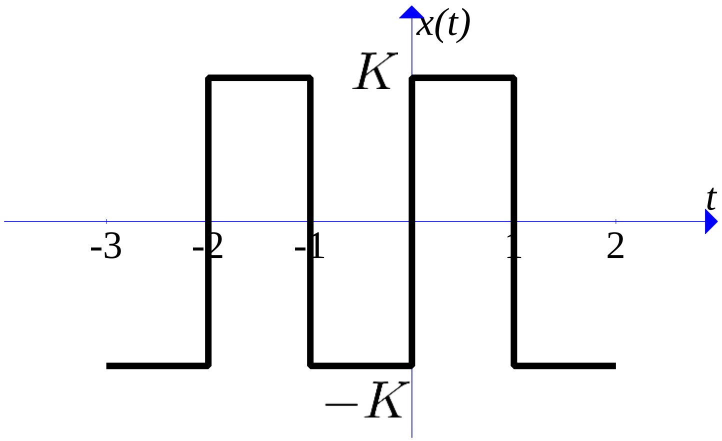
FOURIER SERIES

- **Example**

$$x(t)=\begin{cases}-K, & -1<t<0\\ K, & 0<t<1\end{cases}$$



Rectangle pulses

FOURIER SERIES

- **Amplitude and phase**
  - The Fourier series coefficients are usually complex numbers

$$c_n = a_n + jb_n = |c_n| e^{j\theta_n}$$

  - Amplitude line spectrum: amplitude as a function of $n\Omega_0$

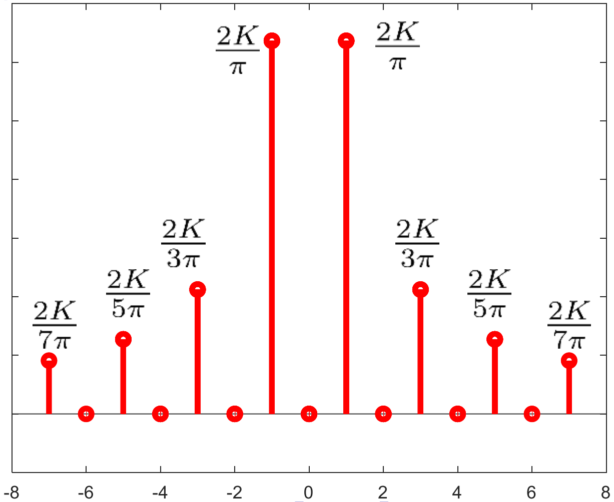
$$|c_n| = \sqrt{a_n^2 + b_n^2}$$

  - Phase line spectrum: phase as a function of $n\Omega_0$

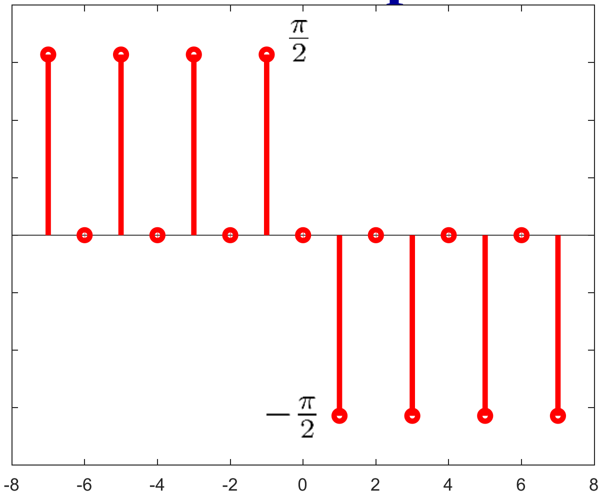
$$\theta_n = \mathrm{a}\tan \frac{b_n}{a_n}$$

FOURIER SERIES: FREQUENCY DOMAIN

- **Signal represented in frequency domain: line spectrum**



amplitude

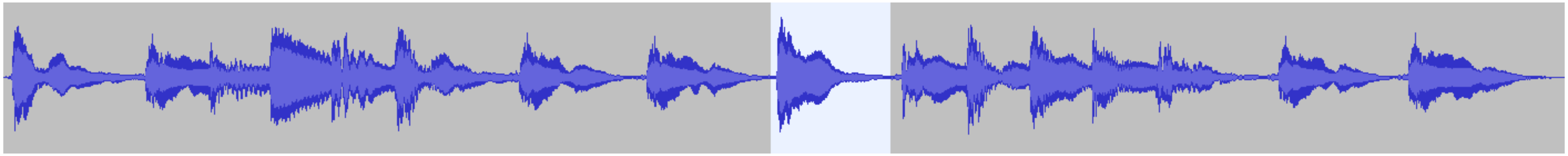


phase

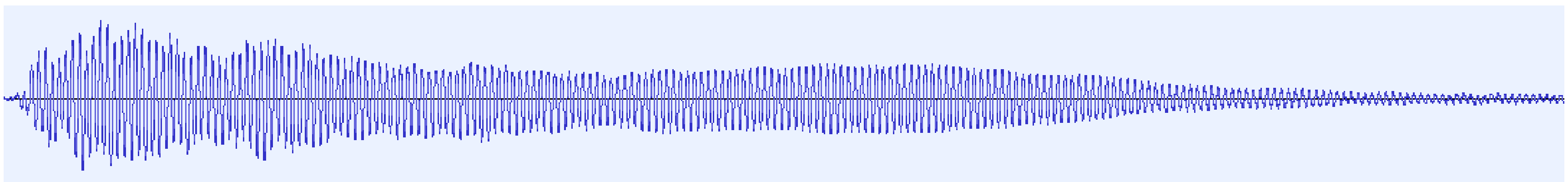
- Each c_n has its own frequency $n\Omega_0$
- The signal is decomposed in frequency domain.
- c_n is called the harmonic of signal $s(t)$ at frequency $n\Omega_0$
- Each signal has many frequency components.
  - The power of the harmonics at different frequencies determines how fast the signal can change.

FOURIER SERIES: FREQUENCY DOMAIN

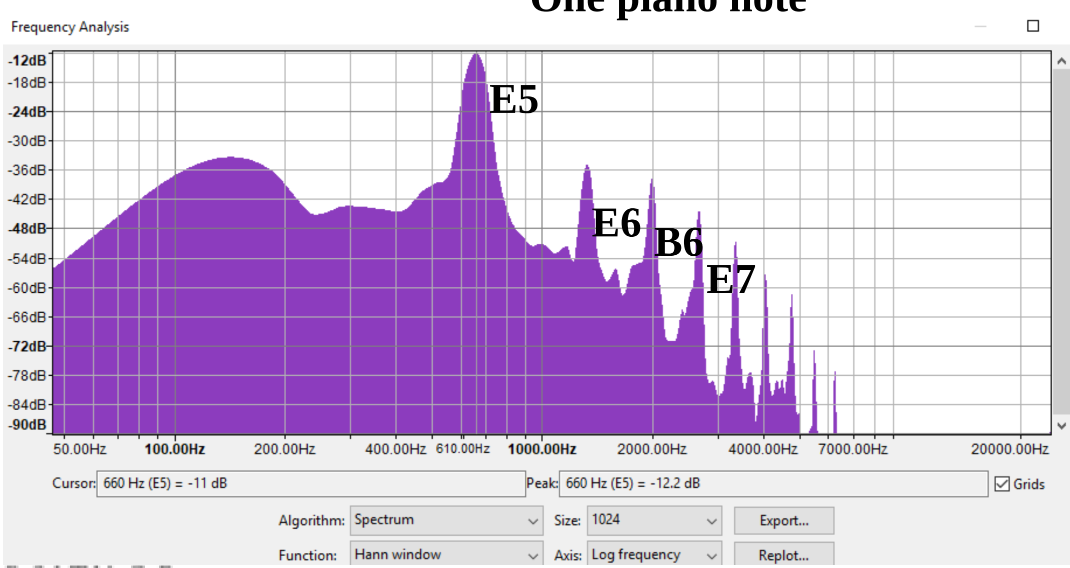
- **Example: Piano Note**



piano notes



One piano note



spectrum

E5: 659.25 Hz
E6: 1318.51 Hz
B6: 1975.53 Hz
E7: 2637.02 Hz

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All graphs in this page are created by using the open-source software Audacity.

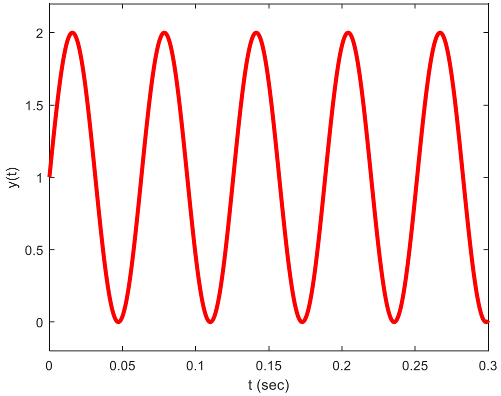
FOURIER SERIES

- **Example**
  - Find the Fourier series of $s(t) = \exp(j\Omega_0 t)$

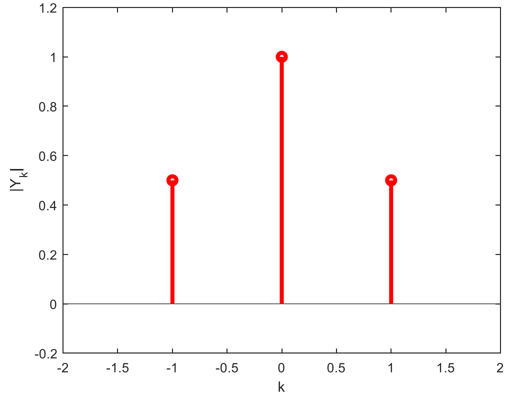
FOURIER SERIES

- **Example**
  - Find the Fourier series of $s(t) = B + A\cos(\Omega_0 t + \theta)$

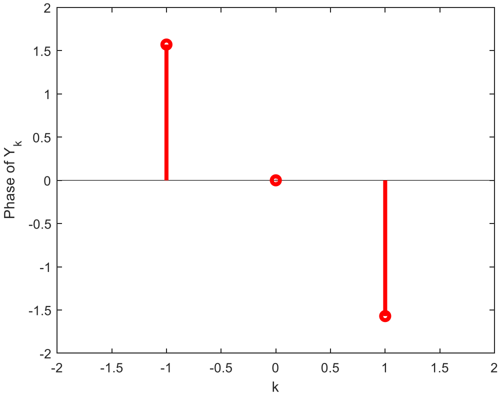
$$y(t) = 1 + \sin(100t)$$



Time domain



Amplitude spectrum

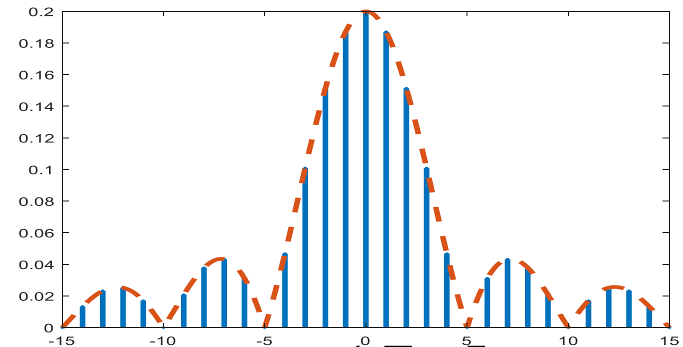


Phase spectrum

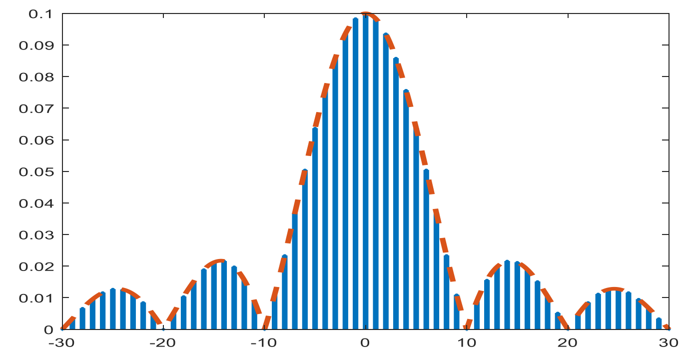
FOURIER SERIES

- **Example**
  - Find the Fourier series of

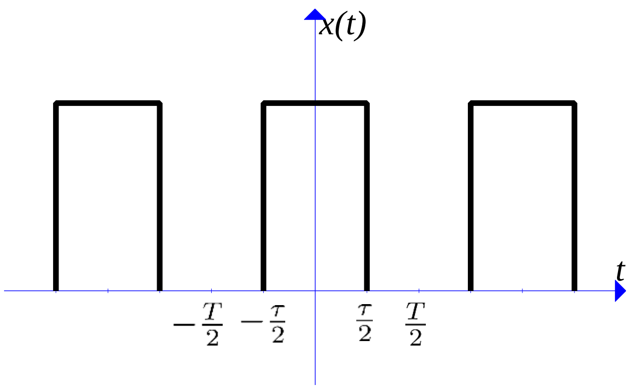
$$s(t)=\begin{cases}0, & -T/2<t<-\tau/2\\ K, & -\tau/2<t<\tau/2\\ 0, & \tau/2<t<T/2\end{cases}$$



$\tau=1, T=5$

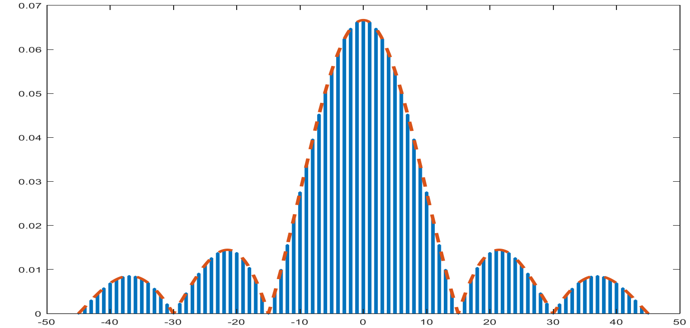


$\tau=1, T=10$



Time domain

$$c_n=\frac{K\tau}{T}\sin c\left(\frac{n\tau}{T}\right)$$



$\tau=1, T=15$

FOURIER SERIES: DIRICHLET CONDITIONS

- **Can any periodic signal be decomposed into Fourier series?**
  - Only signals satisfy Dirichlet conditions have Fourier series
- **Dirichlet conditions**
  - 1. $x(t)$ is absolutely integrable within one period

$$\int_{<T>} |x(t)|\,dt < \infty$$

  - 2. $x(t)$ has only a finite number of maxima and minima.
  - 3. The number of discontinuities in x(t) must be finite.

OUTLINE

- Introduction
- Fourier series
- **Properties of Fourier series**
- Systems with periodic inputs

PROPERTIES: LINEARITY

- **Linearity**
  - Two periodic signals with the same period $T_0 = \dfrac{2\pi}{\Omega_0}$

$$x(t) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{jn\Omega_0 t} \qquad\qquad y(t) = \sum_{n=-\infty}^{+\infty} \beta_n e^{jn\Omega_0 t}$$

  - The Fourier series of the superposition of two signals is

$$k_1 x(t) + k_2 y(t) = \sum_{n=-\infty}^{+\infty} (k_1\alpha_n + k_2\beta_n) e^{jn\Omega_0 t}$$

  - If

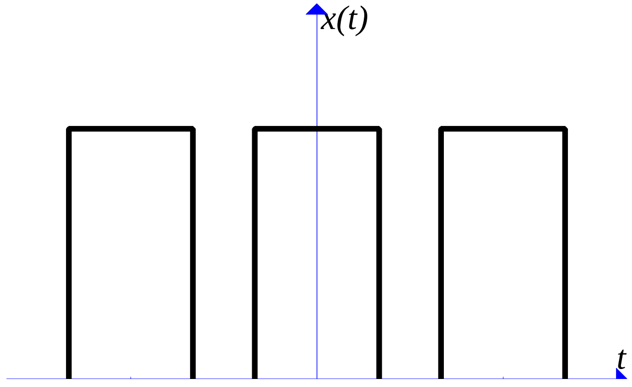
$$x(t) <=> \alpha_n \qquad\qquad y(t) <=> \beta_n$$

    - then

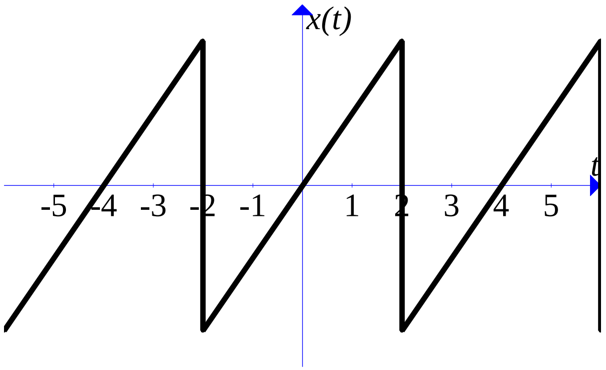
$$k_1 x(t) + k_2 y(t) <=> (k_1\alpha_n + k_2\beta_n)$$

PROPERTIES: EFFECTS OF SYMMETRY

- **Symmetric signals**
  - A signal is even symmetry if: $x(t) = x(-t)$
  - A signal is odd symmetry if: $x(t) = -x(-t)$
  - The existence of symmetries simplifies the computation of Fourier series coefficients.



Even symmetric



Odd symmetric

PROPERTIES: EFFECTS OF SYMMETRY

- **Fourier series of even symmetry signals**
  - If a signal is even symmetry, then

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n \cos(n\Omega_0 t)$$

$$a_n = \frac{2}{T_0}\int_0^{T_0/2} x(t)\cos(n\Omega_0 t)dt$$

- **Fourier series of odd symmetry signals**
  - If a signal is odd symmetry, then

$$x(t) = \sum_{n=1}^{+\infty} b_n \sin(n\Omega_0 t)$$

$$b_n = \frac{2}{T_0}\int_0^{T_0/2} x(t)\sin(n\Omega_0 t)dt$$

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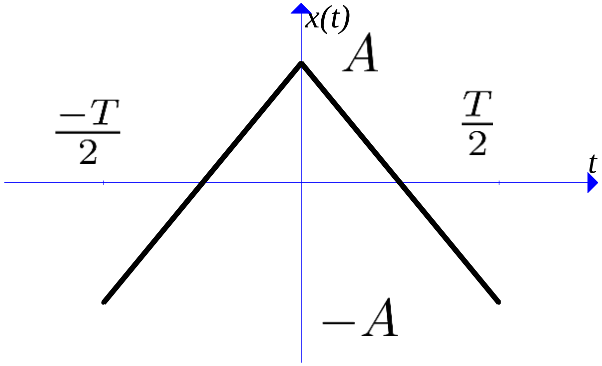
PROPERTIES: EFFECTS OF SYMMETRY

- **Example**

$$x(t)=\begin{cases} A-\dfrac{4A}{T}t, & 0<t<T/2 \\ \dfrac{4A}{T}t-3A, & T/2<t<T \end{cases}$$



Graph of x(t)

PROPERTIES: SHIFT IN TIME

- **Shift in time**
  - If $x(t)$ has Fourier series c_n, then $x(t-t_0)$ has Fourier series $c_n e^{-jn\Omega_0 t_0}$

if $x(t) \Leftrightarrow c_n$ **, then** $x(t-t_0) \Leftrightarrow c_n e^{-jn\Omega_0 t_0}$

  - Proof:

PROPERTIES: PARSEVAL'S THEOREM

- **Review: power of periodic signal**

$$P = \frac{1}{T}\int_0^T |x(t)|^2\, dt$$

- **Parseval's theorem**

if $x(t) \Leftrightarrow \alpha_n$

then $\frac{1}{T}\int_0^T |x(t)|^2\, dt = \sum_{m=-\infty}^{+\infty} |\alpha_m|^2$

  – Proof:

The power of signal can be computed in frequency domain!

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PROPERTIES: PARSEVAL'S THEOREM

- **Example**
  - Use Parseval's theorem find the power of $x(t) = A\sin(\Omega_0 t)$

OUTLINE

- Introduction
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PERIODIC INPUTS: COMPLEX EXPONENTIAL INPUT

- **LTI system with complex exponential input**

$$x(t) = e^{j\Omega t} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) \otimes h(t) = h(t) \otimes x(t)$$

$$= \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau$$

$$= \exp(j\Omega t) \int_{-\infty}^{+\infty} h(\tau) \exp(-j\Omega\tau) d\tau$$

- **Transfer function**

$$H(\Omega) = \int_{-\infty}^{+\infty} h(\tau) \exp(-j\Omega\tau) d\tau$$

  – For LTI system with complex exponential input, the output is

$$y(t) = H(\Omega) \exp(j\Omega t)$$

  – It tells us the system response at different frequencies

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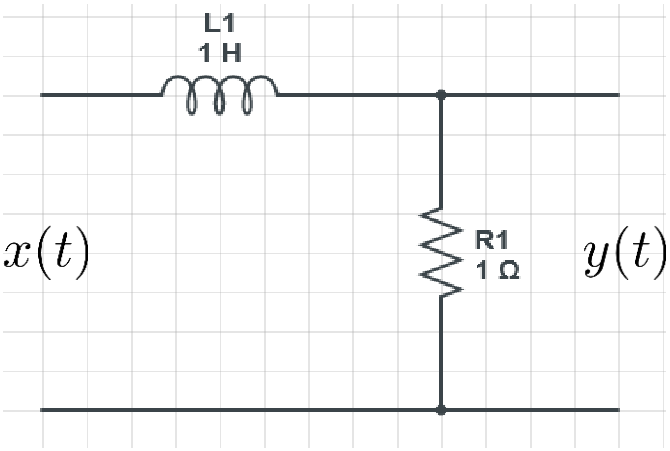
PERIODIC INPUT

- **Example:**
  - For a system with impulse response $h(t) = \delta(t - t_0)$ find the transfer function

PERIODIC INPUT:

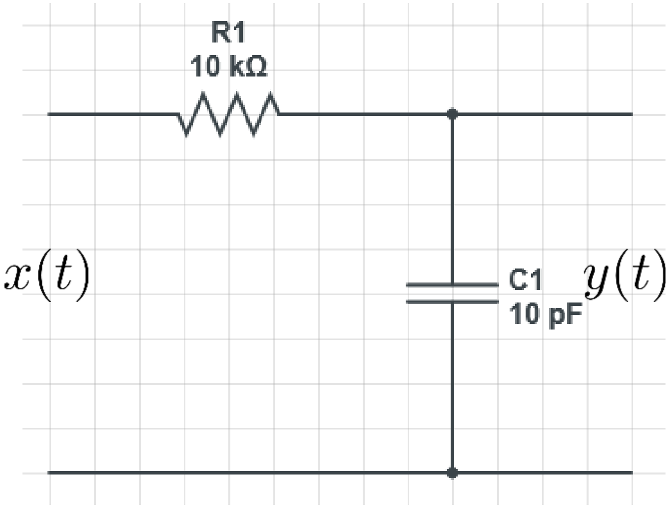
- **Example**
  - Find the transfer function of the system shown in figure.



RL circuit

PERIODIC INPUTS

- **Example**
  - Find the transfer function of the system shown in figure



RC circuit

PERIODIC INPUTS: TRANSFER FUNCTION

- **Transfer function**
  - For system described by differential equations

$$\sum_{i=0}^{n} p_i y^{(i)}(t) = \sum_{i=0}^{m} q_i x^{(i)}(t)$$

$$\boxed{H(\Omega) = \frac{\sum_{i=0}^{m} q_i (j\Omega)^i}{\sum_{i=0}^{n} p_i (j\Omega)^i}}$$

PERIODIC INPUTS

- **LTI system with periodic inputs**
  - Periodic inputs: $x(t) = \sum_{n=-\infty}^{+\infty} c_n \exp(jn\Omega_0 t)$ $\qquad \omega_0 = \frac{2\pi}{T}$

$$e^{jn\Omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow e^{jn\Omega_0 t} H(n\Omega_0)$$

linear:

$$\sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} H(n\Omega_0)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} H(n\Omega_0)$$

For system with periodic inputs, the output is the weighted sum of the transfer function.

PERIODIC INPUTS

- **Procedures:**
  - To find the output of LTI system with periodic input
    - 1. Find the Fourier series coefficients of periodic input $x(t)$.

$$\alpha_n = \frac{1}{T}\int_0^T x(t)e^{-jn\Omega_0 t}dt \qquad \Omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

period of $x(t)$

    - 2. Find the transfer function of LTI system $H(\Omega)$
    - 3. The output of the system is

$$y(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} H(n\Omega_0)$$

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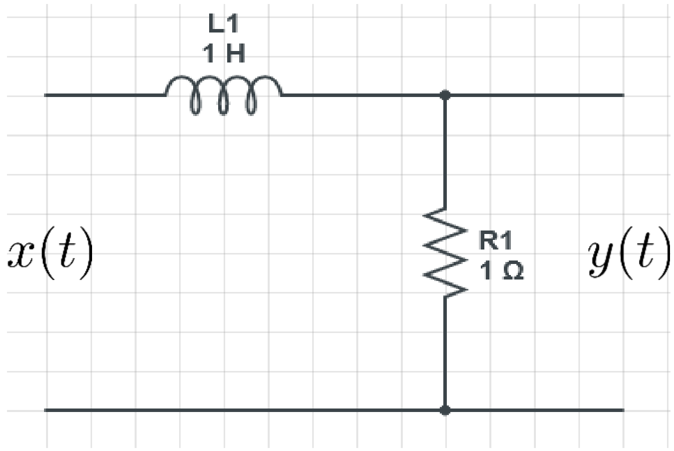
PERIODIC INPUTS

- **Example**
  - Find the response of the system when the input is

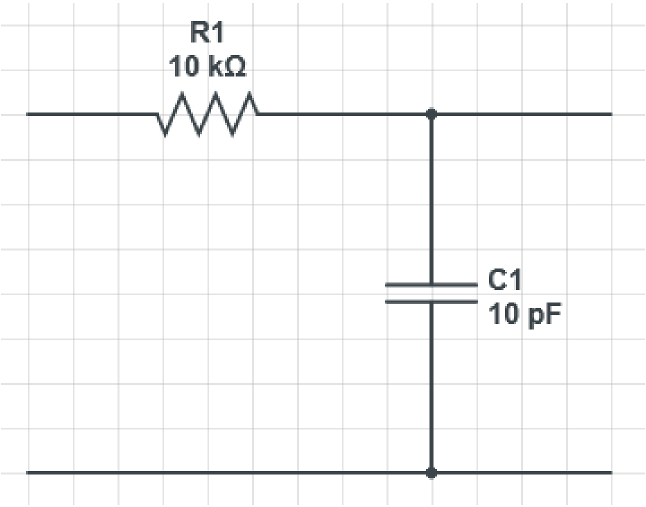
$$x(t) = 4\cos(t) - 2\cos(2t)$$



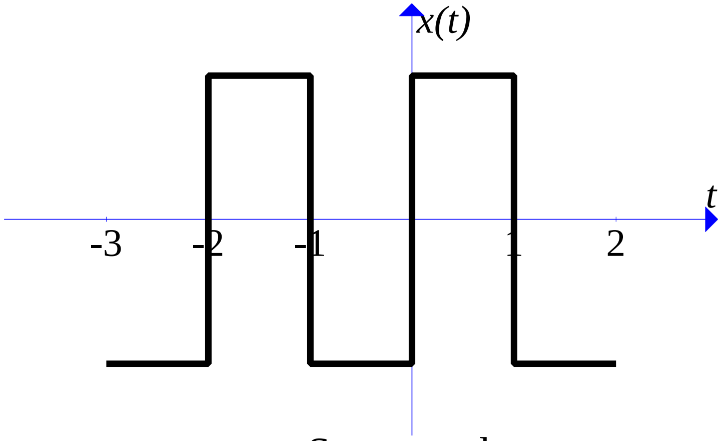
RL Circuit

PERIODIC INPUTS

- **Example**
  - Find the response of the system when the input is shown in figure.



RC circuit



Square pulses

PERIODIC INPUTS: GIBBS PHENOMENON

- **The Gibbs Phenomenon**
  - Most Fourier series has infinite number of elements→ unlimited bandwidth

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

- What if we truncate the infinite series to finite number of elements?

$$x_N(t) = \sum_{n=-N}^{+N} c_n e^{jn\Omega_0 t}$$

  - The truncated signal, $x_N(t)$, is an approximation of the original signal $x(t)$

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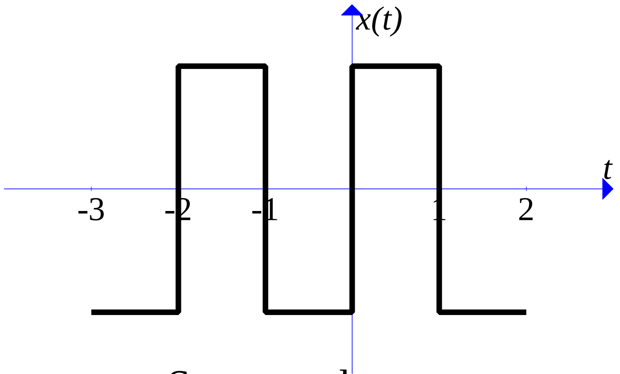
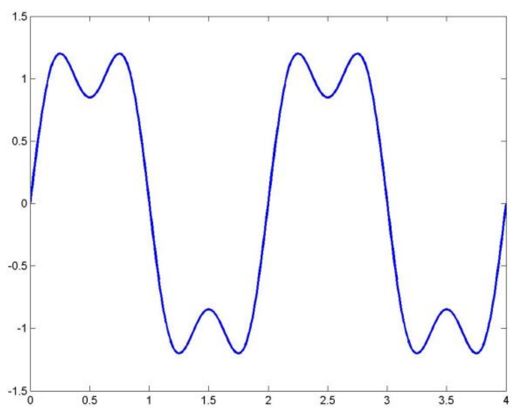
PERIODIC INPUTS: GIBBS PHENOMENON



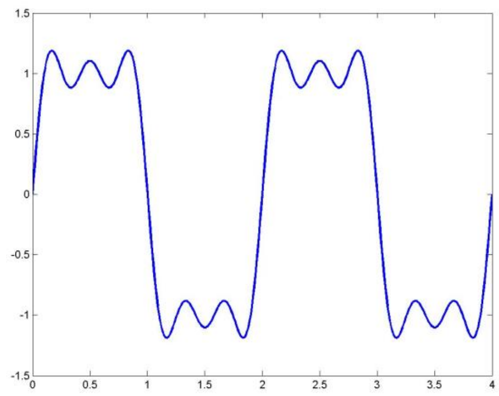
Square pulses

$$c_n = \begin{cases} \dfrac{2K}{j\pi}\dfrac{1}{n}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

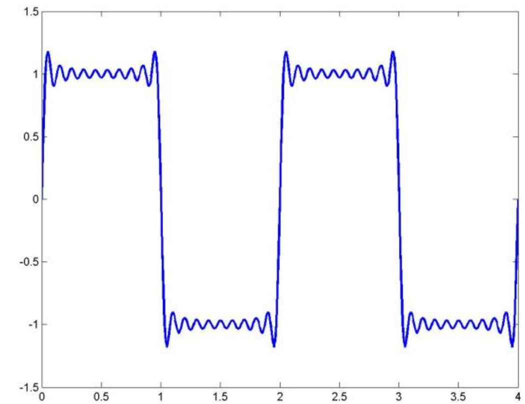
$$x_N(t) = \sum_{n=-N}^{+N} c_n e^{jn\Omega_0 t}$$



$x_3(t)$



$x_5(t)$



$x_{19}(t)$

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FOURIER SERIES

- **Analogy: Optical Prism**
  - Each color is an Electromagnetic wave with a different frequency

Optical prism