Connectivity Analysis of Mobile Linear Networks with Delay Constraint

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Abstract— **The connectivity properties of a mobile linear network with high speed mobile nodes, dynamic node population, and a strict delay constraint are investigated. A new mobility model is developed to represent the steady state node distributions in terms of random node location and random node population. The model accurately captures the statistical properties of random node arrival, time-varying node speed, and the distinct behaviors of nodes following different traffic patterns. With the new mobility model, the statistical properties of network connectivity are studied and identified. Unlike most previous works that do not consider the impacts of transmission latency, which is critical for real time applications, this paper identifies the quantitative relationship between network connectivity and delay constraint by placing a bound on the maximum transmission distance. The connectivity analysis is performed with a novel geometry-assisted analytical method. Exact connectivity probability expressions are developed by using the volumes of a hypercube intersected by a hyperplane. Results presented in this paper provide insight on the design and operations of mobile linear networks, such as a vehicular ad hoc network (VANET) that has rapidly changing network topology and node density.**

I. INTRODUCTION

Network connectivity is a critical metric for the planning, design, and evaluation of ad hoc networks. Two nodes in an ad hoc network are connected if they can exchange information with each other, either directly or indirectly, within certain latency constraint. In many practical networks, such as vehicular ad hoc network (VANET) [1], all nodes are constantly moving at high speeds, and this results in a rapidly changing network topology with a dynamic node population. If the network is delay tolerant, then node mobility can improve network connectivity by utilizing a store-and-forward (SF) scheme, which allows intermediate nodes to temporarily store information and deliver it at a different location [2], [3]. On the other hand, most practical networks have strict constraint on transmission latency, which requires intermediate nodes immediately forward received information to the next hop. Such a scheme is denoted as receive-and-forward (RF) in this paper. Therefore, it's essential to identify the impacts of key network parameters, such as node mobility, delay constraint, and transmission power (or transmission range), on the connectivity probability of ad hoc networks.

The study of network connectivity has attracted considerable interests recently [4] – [10]. Most of the connectivity analyses were performed for networks with randomly distributed

stationary nodes [4] – [6]. There are limited works on the connectivity of network with mobile nodes [1], [3], [7] – [10]. The critical transmission range in a sparse mobile network is studied in [7] with computer simulations. Hybrid numerical-simulation analysis is performed in [8] to identity the approximated connectivity of an arbitrary two-dimensional network with stationary and mobile nodes. The exact connectivity probability of two nodes with distance l on a linear network modeled by a Poisson point process is presented in [9], and the result was extended in [10] by considering interference.

Most of the aforementioned works employ the RF scheme, where the main sources of delay arise from the processing time at intermediate nodes and the propagation delay of the electromagnetic waveform. As a result, the one-way transmission delay is directly related to the source-destination distance. To meet the strict latency constraint of real time applications, it is necessary to limit the maximum distance and/or number of hops involved during information transmission, and such a limit has significant impact on the connectivity of a mobile network. To the best of our knowledge, there is no works in the literature devoted to the identification of mobile network connectivity under delay constraint.

This paper focuses on the analysis of network connectivity for a linear network with high speed mobile nodes, dynamic node population, and strict delay constraint. A new mobility model is developed to represent the steady state node distribution of a mobile linear network with the tools and theories from M/G/∞ queuing systems [11]. It's more comprehensive and practical compared to the stationary network models used by most previous works. Based on the new mobility model, the statistical properties of the connectivity of mobile linear networks with a delay constraint are studied. The connectivity analysis is performed with the assistance of an innovative geometry-assisted analytical method. Specifically, the volume of an n-dimensional convex polytope, which is obtained by intersecting a hypercube (n-cube) with a hyperplane, is derived. The geometric results significantly simplify the connectivity analysis, and they lead to exact closed-form network connectivity probability expressions.

II. SYSTEM MODEL AND PRELIMINARY STATISTICS

A. Mobility Model

Consider a section of a unidirectional network defined by the interval $\mathcal{L} = [0, L]$. Each node enters the network at $x = 0$, and exits at $x = L$. The mobility of the nodes in the network are modeled after the following assumptions.

A.1) Nodes are divided into I classes corresponding to vehicles on different lanes of a highway. The nodes belonging to the same class share independently and identically distributed (i.i.d.) mobility properties.

A.2) A class i node enters the VANET following a Poisson distribution with arrival rate λ_{0i}, $i = 1, \cdots, I$, [9] and [10].

A.3) The time that a class i node spends on a section of the highway, $[x_0, x_0 + x]$, is a random variable (RV), $T_i(x)$, with mean proportional to the section length x, *i.e.*,

$$\mu_{T i(x)} = \int_0^\infty \tau f_{T_i(x)}(\tau) d\tau = \frac{x}{\mu_i}, \tag{1}$$

where $f_{T_i(x)}(\tau)$ is the probability density function (pdf) of $T_i(x)$, and μ_i is a scaling factor related to the distribution of the speed of a class i node.

A.4) Nodes can freely pass each other.

With the above mobility assumptions, at any moment t, the number of nodes inside $[0, L]$, $N(t)$, and the location of a given node inside $[0, L]$, $X(t)$, are random variables. We are interested in the steady state distribution of $N(t)$ and $X(t)$ as $t \to \infty$, and the results are summarized in the following Lemma.

Lemma 1: Consider a length-L linear network with node mobility described in Assumptions A.1) - A.4). At the steady state $(t \to \infty)$, the number of nodes in $[0, L]$ follows a Poisson distribution with parameter $\lambda = L \sum_{i=1}^I \frac{\lambda_{0i}}{\mu_i}$, and these nodes are independently and uniformly distributed inside $[0, L]$.

Proof: 1) Steady state distribution of node population. Based on the mobility assumptions, each class of nodes can be modeled as an $M/G/\infty$ queuing system [11]. Let $N_i(t)$ denote the number of class-i nodes inside $[0, L]$ at time t, then mapping the analysis of $M/G/\infty$ queue, we have

$$P\{N_i(t)=n\}=\sum_{k=n}^\infty P\{N_i(t) = n|K_i(t) = k\} P\{K_i(t) = k\}, \tag{2}$$

where $K_i(t)$ is the number of class-i nodes that arrives during $[0, t]$, and it follows a Poisson distribution with parameter $\lambda_{0i}t$.

Since a node is either inside or outside $[0, L]$, $N_i(t)$ conditioned on $K_i(t)$ follows a binomial distribution with parameter $\alpha_i(t)$, which is defined as the probability that a node that arrived between $[0, t]$ is still within $[0, L]$ at time t. For a node arriving between $[0, t]$, $\alpha_i(t) = \int_0^t P\{T_i(L) \geq t - \tau\} f_{t_0|t}(\tau) d\tau$, where $T_i(L)$ is the amount of time that the node spends on $[0, L]$, and $f_{t_0|t}(\tau)$ is the conditional pdf of the node's arrival time, t_0, given the fact that the node arrived between $[0, t]$. For Poisson arrival, $f_{t_0|t}(\tau) = \frac{1}{t}$, for $0 \leq \tau \leq t$ [12, Theorem 5.2]. Let $F_{T_i(L)}(\tau)$ denote the cumulative distribution function (cdf) of $T_i(L)$, we have

$$\alpha_i(t) = \frac{1}{t} \int_0^t \left[1 - F_{T_i(L)}(\tau)\right] d\tau. \tag{3}$$

Substituting the binomial distribution, $P(N_i(t) = n|K_i(t) = k)$, into (2) leads to $P\{N_i(t) = n\} = \frac{\lambda_i^n(t)}{n!} e^{-\lambda_i(t)}$, which is a Poisson distribution with

parameter $\lambda_i(t) = \lambda_{0i} \int_0^t \left[1 - F_{T_i(L)}(\tau)\right] d\tau$. Thus $N_i \triangleq \lim_{t\to\infty} N_i(t)$ is Poisson distributed with parameter $\lambda_i \triangleq \lim_{t\to\infty} \lambda_i(t) = \frac{\lambda_{0i}L}{\mu_i}$. Since the sum of independent Poisson RVs is still Poisson distributed [12], the total number of nodes inside $[0, L]$, $N = \sum_{i=1}^I N_i$, is a Poisson RV with parameter $\lambda = L \sum_{i=1}^I \frac{\lambda_{0i}}{\mu_i}$.

2) Steady state distribution of node location. Let $X_i(t)$ denote the location of a class-i node at time t, then

$$P\{X_i(t) < x\} = \frac{1}{t} \int_0^t \left[1 - F_{T_i(x)}(t - \tau)\right] d\tau. \tag{4}$$

Let X_i denote the location of a class-i node inside $[0, L]$ as $t \to \infty$, then for any interval $[a, b] \subseteq [0, L]$, the probability $P\{a \leq X_i \leq b\} = \lim_{t\to\infty} P\{a \leq X_i(t) \leq b|0 \leq X_i(t) \leq L\}$ can be written as

$$P\{a \leq X_i \leq b\} = \frac{P\{X_i \leq b\} - P\{X_i \leq a\}}{P\{0 \leq X_i \leq L\}} = \frac{b - a}{L}, \tag{5}$$

where the second equality is obtained from (1) and (4). Since the above probability is true for all $[a, b] \subseteq [0, L]$, $X_i \sim U([0, L])$. ∎

B. Preliminary Statistics

Let X_m, for $m = 1, 2, \cdots, n$, denote the position of n independent nodes uniformly distributed over the interval $[0, L]$, *i.e.*, $X_m \sim U([0, L])$. The number of nodes, $N = n$, is a Poisson RV with parameter $\lambda = L \sum_{i=1}^I \frac{\lambda_{0i}}{\mu_i}$.

Ordering the n RVs in ascending order yields a group of new RVs, $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. The joint distribution of the ordered RVs is given in the following Lemma [14].

Lemma 2: Define the ordered vector of i.i.d. uniform RVs as $\mathbf{X}_{(o)} = [X_{(1)}, X_{(2)}, \cdots, X_{(n)}]^T$, where $\mathbf{A}^T$ denotes matrix transpose, then the pdf of $\mathbf{X}_{(o)}$ can be written as $f_{\mathbf{X}_{(o)}}(x_1, x_2, \cdots, x_n) = \frac{n!}{L^n}$, for $0 \leq x_1 \leq \cdots \leq x_n \leq L$. ∎

The study of node connectivity requires the investigation of the distribution of the distance between node pairs. Define the size $(n - 1) \times 1$ distance vector as $\mathbf{Y} = [Y_1, Y_2, \cdots, Y_{n-1}]^T$, with $Y_m = X_{(m+1)} - X_{(m)}$. To facilitate analysis, denote $Y_0 = X_{(1)}$. Prefixing $\mathbf{Y}$ with Y_0 leads to an extended distance vector, $\tilde{\mathbf{Y}} = [Y_0, Y_1, \cdots, Y_{n-1}]^T$. The distribution of $\tilde{\mathbf{Y}}$ is presented in the following Lemma.

Lemma 3: The pdf of the extended distance random vector, $\tilde{\mathbf{Y}}$, is $f_{\tilde{\mathbf{Y}}}(y_0, y_1, \cdots, y_{n-1}) = \frac{n!}{L^n}$, if $\sum_{m=0}^{n-1} y_m \leq L$ and $0 \leq y_m \leq L$, for $m = 0, \cdots, n-1$, and $f_{\tilde{\mathbf{Y}}}(y_0, y_1, \cdots, y_{n-1}) = 0$ otherwise.

Proof: The proof is in Appendix A. ∎

III. GEOMETRIC RESULTS

The network connectivity will be investigated with a new geometry-assisted analytical method, which translates the derivation of the connectivity probabilities to the evaluation of the volume of an n-dimensional convex polytope, $\mathcal{D}_n(d, L) = \{\mathbf{y}_n | \sum_{m=1}^n y_m \leq L, \quad 0 \leq y_m \leq d\}$. The volume, $\text{Vol}[\mathcal{D}_n(d, L)] = \int \cdots \int_{\mathbf{y}_n \in \mathcal{D}_n(d, L)} d\mathbf{y}_n$, is evaluated in this section.

Theorem 1: The volume of the polytope, $\mathcal{D}_n(d, L)$, which is obtained by intersecting a hypercube, $\mathcal{C}_n(d) = \{\mathbf{y}_n \mid 0 \leq y_m \leq d, m = 1, \cdots, n\}$, with a hyperplane, $\mathcal{P}_n(L) = \{\mathbf{y}_n \mid \sum_{m=1}^{n} y_m = L\}$, is

$$\mathrm{Vol}\,[\mathcal{D}_n(d, L)] = V_n^k(d, L) \triangleq \frac{1}{n!} \sum_{m=0}^{k} (-1)^m \binom{n}{m} (L - md)^n,$$

$$\text{if } \bar{d} \in \mathcal{L}_k(n), \text{ for } k = 0, 1, \cdots, n, \quad (6)$$

where $\bar{d} = d/L$, and $\mathcal{L}_k(n) = \left[\frac{1}{k+1}, \frac{1}{k}\right)$ for $k = 1, \cdots, n-1$, and $\mathcal{L}_n(n) = \left[0, \frac{1}{n}\right)$.

Proof: Proof by induction. The proof for $n = 1$ is trivial. Assume (6) is true for $\mathrm{Vol}\,[\mathcal{D}_{n-1}(d, L)]$. The induction part of the proof is divided into three cases, $\bar{d} \in \mathcal{L}_0(n)$, $\bar{d} \in \mathcal{L}_k(n)$, for $k = 1, \cdots, n-1$, and $\bar{d} \in \mathcal{L}_n(n)$.

1) $\bar{d} \in \mathcal{L}_k(n)$, for $k = 1, \cdots, n-1$. Based on the volume definition, when $d_0 \in \mathcal{L}_k(n)$, we have

$$V_n^k(d, L) = \int_0^d \mathrm{Vol}\,[\mathcal{D}_{n-1}(d, L - y_n)]\, dy_n,$$

$$= \int_{kd}^{L} V_{n-1}^k(d, z)dz + \int_{L-d}^{kd} V_{n-1}^{k-1}(d, z)dz, \quad (7)$$

where $z = L - y_n$ is used in the second equality. The partition of the integration interval is performed based on the definition interval of $V_n^k(d, L)$. Solving the two integrals in (7) with the definition of $V_n^k(d, L)$ and simplifying lead to

$$V_n^k(d, L) = \frac{1}{n!} \left\{ L^n + \sum_{m=1}^{k} (-1)^m \binom{n}{m} (L - md)^n \right\}, \quad (8)$$

where the identity

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}, \quad (9)$$

is used in the simplification. Eqn. (8) simplifies to (6).

2) $\bar{d} \in \mathcal{L}_n(n)$. The volume can be directly written as

$$V_n^n(d, L) = \int_0^d V_{n-1}^{n-1}(d, L - y_n)dy_n, \quad (10)$$

due to the fact that $0 \leq \frac{d}{L - y_n} < \frac{1}{n-1}$ in the entire integration interval $y_n \in [0, d]$. Simplifying the above equation with the definition of $V_{n-1}^{n-1}(d, L)$, and the combinatorics identities, $\binom{n-1}{0} = \binom{n}{0}$, and (9), lead to (6).

3) $\bar{d} \in \mathcal{L}_0(n)$. The condition $1 \leq \bar{d}$ means $0 < y_n \leq \sum_{m=1}^{n} y_m \leq L \leq d$. Thus the integration limit of y_n is $[0, L]$. In this case, $V_n^0(d, L)$ can be calculated as

$$V_n^0(d, L) = \int_0^L V_{n-1}^0(d, L - y_n)dy_n. \quad (11)$$

which can be simplified to $V_n^0(d, L) = \frac{1}{n!}L^n$.

Since the induction is true for $\mathrm{Vol}\,[\mathcal{D}_n(d, L)]$ over all the definition intervals, the Theorem is proved. ∎

IV. CONNECTIVITY OF MOBILE LINEAR NETWORK

In this section, the connectivity properties of a delay constrained mobile linear network are investigated with the help of the geometric result.

Before moving on to the connectivity analysis, we first establish an analytical bound that identifies the quantitative relationship between delay constraint and node distance for networks employing the receive-and-forward scheme. The result is presented in the following Lemma.

Lemma 4: Consider two connected nodes with distance $md < l \leq (m + 1)d$, where d is the transmission range of a node. During receive-and-forward, if the next hop is chosen as the node that is the furthest node within the transmission range of the current node, then the one way transmission delay between the two nodes is bounded by

$$l\left(\frac{t_p}{d} + \frac{1}{c}\right) - t_p \leq t_d < l\left(2\frac{t_p}{d} + \frac{1}{c}\right), \quad (12)$$

where t_p is the processing time at one intermediate node, and c is the speed of light.

Proof: The proof is in Appendix B. ∎

The result in Lemma 4 indicates that the transmission delay is proportional to the source-destination distance. Thus a certain delay constraint can be achieved by limiting the source-destination distance. Given a delay constraint $t_{\max}$, two nodes with distance l satisfying $l \leq l_{\max} \triangleq t_{\max}/\left[\left(2\frac{t_p}{d} + \frac{1}{c}\right)\right]$ is guaranteed to have $t_d < t_{\max}$. In the following analysis, we will study the impacts of delay constraint $t_{\max}$ by limiting the maximum distance between node pairs with $l_{\max}$.

A. Connectivity of an n-node network

We first study the connectivity of a linear network with a fixed number of n nodes, and the results will be used to assist the analysis of networks with dynamic number of nodes.

Theorem 2: For a linear network with n nodes uniformly distributed over a section with length L, if the maximum transmission range of each node is d, and the maximum distance between any two nodes is bounded by $l_{\max}(\geq d)$ due to delay constraint, then the probability that all the n nodes are connected is

$$P_n(\bar{d}, \bar{l}_{\max}) = \begin{cases} \sum_{m=0}^{k}(-1)^m \binom{n-1}{m}\left[(\bar{l}_{\max} - m\bar{d})^n + \right. \\ \left. n(1 - \bar{l}_{\max})(\bar{l}_{\max} - m\bar{d})^{n-1}\right], & \bar{d} \leq \bar{l}_{\max} \leq 1 \\ \sum_{m=0}^{j}(-1)^m \binom{n-1}{m}\left(1 - m\bar{d}\right)^n, & \bar{l}_{\max} > 1 \end{cases}$$

where $\bar{l}_{\max} = l_{\max}/L$, $\bar{d}/\bar{l}_{\max} \in \mathcal{L}_k(n-1)$, $\bar{d} \in \mathcal{L}_j(n-1)$, for $k, j = 1, \cdots, n-1$.

Proof: When $\bar{d} \leq \bar{l}_{\max} \leq 1$, the probability can be expressed as $P_n(\bar{d}, \bar{l}_{\max}) = P\{\mathbf{Y} \in \mathcal{D}_{n-1}(d, l_{\max})\}$, which can be alternatively written by

$$P_n(\bar{d}, \bar{l}_{\max}) = \underbrace{P\{Y_0 \in [0, L - l_{\max}), \mathbf{Y} \in \mathcal{D}_{n-1}(d, l_{\max})\}}_{P_1} +$$

$$\underbrace{P\{Y_0 \in [L - l_{\max}, L - d), \mathbf{Y} \in \mathcal{D}_{n-1}(d, L - Y_0)\}}_{P_2} + \underbrace{P\{Y_0 \in [L - d, L]\}}_{P_3}.$$

The inclusion of Y_0 in the above expression enables the utilization of $\tilde{\mathbf{Y}}$, which has a constant valued pdf. A constant valued pdf allows the application of the volume result for the connectivity analysis.

The probability P_1 can be expressed as

$$P_1 = \frac{n!}{L^n}(L - l_{\max})\text{Vol}\left[\mathcal{D}_{n-1}(d, l_{\max})\right]. \tag{13}$$

Similarly, the probability P_2 can be expressed as

$$P_2 = \frac{n!}{L^n}\int_{L-l_{\max}}^{L-d}\text{Vol}\left[\mathcal{D}_{n-1}(d, L - y_0)\right]dy_0. \tag{14}$$

Since $\text{Vol}\left[\mathcal{D}_{n-1}(d, L - l_{\max})\right]$ assumes different expressions when $\frac{d}{L-y_0}$ falls in different definition intervals, the integration limit of (14) needs to be partitioned into several sections as $[L - l_{\max}, L - d] = [L - l_{\max}, L - kd] \cup \left(\cup_{m=1}^{k-1}[L - (m+1)d, L - md)\right)$. With such a partition, the probability in (14) can be written as

$$P_2 = \frac{n!}{L^n}\left[\int_{L-l_{\max}}^{L-kd}V_{n-1}^k(d, L - y_0)dy_0 + \right.$$
$$\left. \sum_{m=1}^{k-1}\int_{L-(m+1)d}^{L-md}V_{n-1}^m(d, L - y_0)dy_0\right], \frac{\bar{d}}{\bar{l}_{\max}} \in \mathcal{L}_k(n-1).$$

Solving the two integrals and simplifying lead to

$$P_2 = \sum_{m=0}^{k}(-1)^m\binom{n-1}{m}\left(\bar{l}_{\max} - m\bar{d}\right)^n - \bar{d}^n, \frac{\bar{d}}{\bar{l}_{\max}} \in \mathcal{L}_k(n-1).$$

The probability, P_3, can be calculated as

$$P_3 = \int_{L-d}^{L}f_{Y_0}(y_0)dy_0 = \bar{d}^n, \tag{15}$$

where the pdf, $f_{Y_0}(y_0) = \frac{n}{L^n}(L - y_0)^{n-1}$ [12], is used in the second equality. Combining P_1, P_2 and P_3 in the above equations leads to the first equality in (13).

When $\bar{l}_{\max} > 1$, the probability can be expressed as $P_n(\bar{d}, \bar{l}_{\max}) = P\{\mathbf{Y} \in \mathcal{D}_{n-1}(d, L)\} = P_n(\bar{d}, \bar{l}_{\max} = 1)$, which leads to the second equality in (13). ∎

The connectivity probability is obtained by assuming that there is communication between all the node pairs in the network. If some node pairs in the network do not need to exchange information, then the probability can be considered as a lower bound. In addition, the result presented in Theorem 2 can also be considered as the connectivity probability of a stationary network with n nodes uniformly distributed over a linear section of the network.

B. Connectivity of Network with Dynamic Node Population

With the mobility model presented in Section II, the number of nodes in the network at steady state can be modeled as a Poisson RV with parameter λ as illustrated in Lemma 1.

Before proceeding to the connectivity probability of a mobile network with random number of nodes, we have the following Lemma that will be used during the connectivity analysis

Lemma 5:

$$\sum_{m=n+1}^{+\infty}\binom{m-1}{n}\frac{x^m}{m!} = -(-1)^n\frac{1}{n!}\gamma(n+1, -x), \tag{16}$$

where $\gamma(n, x) = \int_0^x t^{n-1}e^{-t}dt$ is the lower incomplete Gamma function.

Proof: The proof is in Appendix C. ∎

Theorem 3: Consider a linear network of length L. The number of nodes in the network follows a Poisson distribution with parameter λ, and all the nodes distribute uniformly over $[0, L]$. The maximum transmission range of each node is d. Due to delay constraint, two nodes with distance larger than $l_{\max}$ are considered as disconnected. The probability that all the nodes in the network are connected is shown in (17) at the top of the next page. In (17), $\bar{d}/\bar{l}_{\max} \in \mathcal{L}_k(\infty)$, $\bar{d} \in \mathcal{L}_j(\infty)$, $k, j = 1, \cdots, \infty$, and $\mathcal{L}_k(\infty) = \left[\frac{1}{k+1}, \frac{1}{k}\right)$.

Proof: If $\frac{1}{k+1} \leq \frac{d}{l_{\max}} < \frac{1}{k}$, then the connectivity probability can be expressed as $P_\lambda(\bar{d}, \bar{l}_{\max}) = \sum_{m=0}^{\infty}P_m(\bar{d}, \bar{l}_{\max})P\{N = m\}$, which can be further written as

$$P_\lambda(\bar{d}, \bar{l}_{\max}) = e^{-\lambda} + e^{-\lambda}\sum_{m=1}^{\infty}\sum_{n=0}^{\min(m-1,k)}(-1)^n\binom{m-1}{n} \times$$
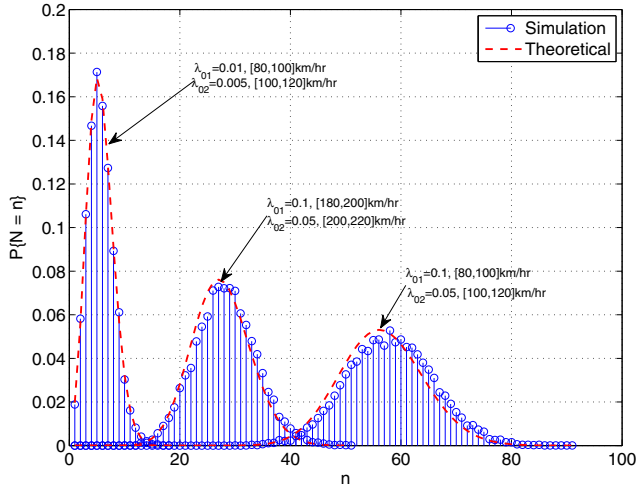$$\left[\frac{\beta_n^m}{m!} + (1 - \bar{l}_{\max})\lambda\frac{\beta_n^{m-1}}{(m-1)!}\right], \tag{18}$$

where $\beta_n = \left(\bar{l}_{\max} - n\bar{d}\right)\lambda$, and $P_0(\bar{d}) = 1$ is used in the second equality. Exchanging the order of summation in (18), and noting the fact that $n \leq k < m$, we can obtain (17) by using the result from Lemma 5. ∎

The result in Theorem 3 is the steady state connectivity probability of a linear network with high speed nodes and dynamic node population. It can be interpreted as the percentage of time that the network is fully connected, or not isolated.

V. Performance Evaluation

We first investigate the accuracy of the mobility model. Fig. 1 shows the steady state distribution of node population and node location under various configurations. In this example, the random nodes are divided into two classes. The speed of a class-i node is modeled as a stationary random process uniformly distributed in $[a_i, b_i]$. It can be easily shown that this implementation satisfies Assumption A.3) with parameter $\mu_i = \frac{b_i - a_i}{\log_e b_i - \log_e a_i}$. The parameters, λ_{0i} and $[a_i, b_i]$, of each class are shown in the figure. The length of the section is 10 km. The analytical curves are generated by using the mobility model presented in Lemma 1. The simulation results are generated by using a discrete event simulator written in Matlab. The steady state simulation results are collected after 10,000 seconds in simulation time after the start of the simulation. Comparison between the simulation results and the theoretical results reveals that the mobility model renders an accurate representation of the steady state distribution of the random node population and node location. In addition, the

$$P_\lambda(\bar{d},\bar{l}_{\max}) = \begin{cases} e^{-\lambda} - \sum_{n=0}^{k} \frac{e^{-\lambda}}{n!} \left[\gamma(n+1, \bar{d}n\lambda - \bar{l}_{\max}\lambda) - (1-\bar{l}_{\max})\lambda^{n+1} \left(n\bar{d} - \bar{l}_{\max}\right)^n e^{\bar{l}_{\max}\lambda - \bar{d}n\lambda} \right], & \bar{l}_{\max} \leq 1, \\ e^{-\lambda} - \sum_{n=0}^{j} \frac{e^{-\lambda}}{n!} \gamma(n+1, \bar{d}n\lambda - \lambda), & \bar{l}_{\max} > 1, \end{cases} \quad (17)$$

(a) Distribution of number of nodes
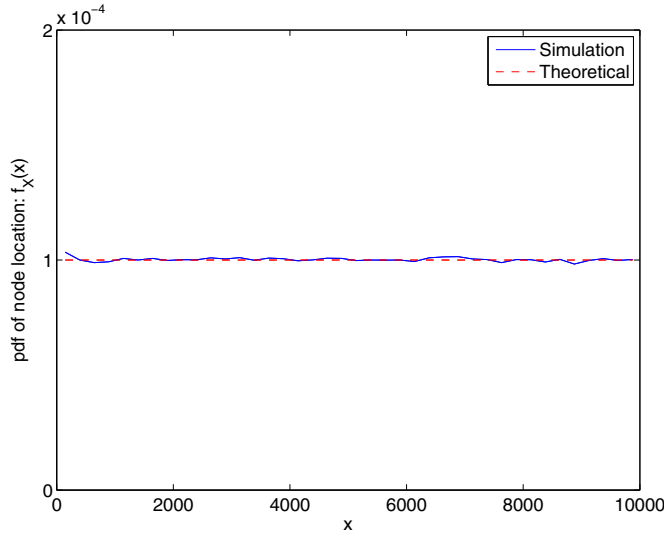
(b) Distribution of node location

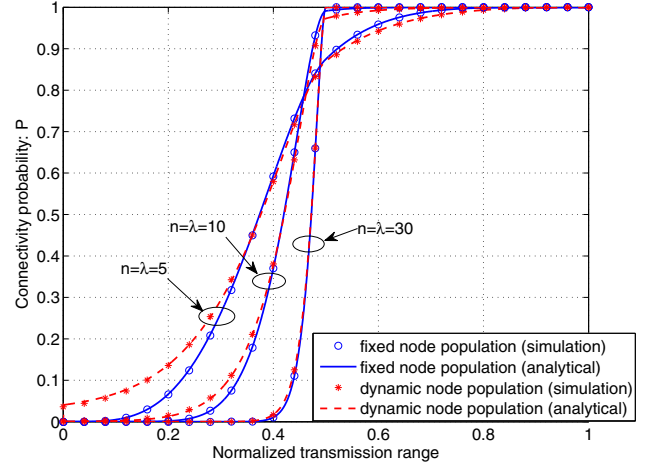Fig. 1. Verification of mobility model as $t \to \infty$.

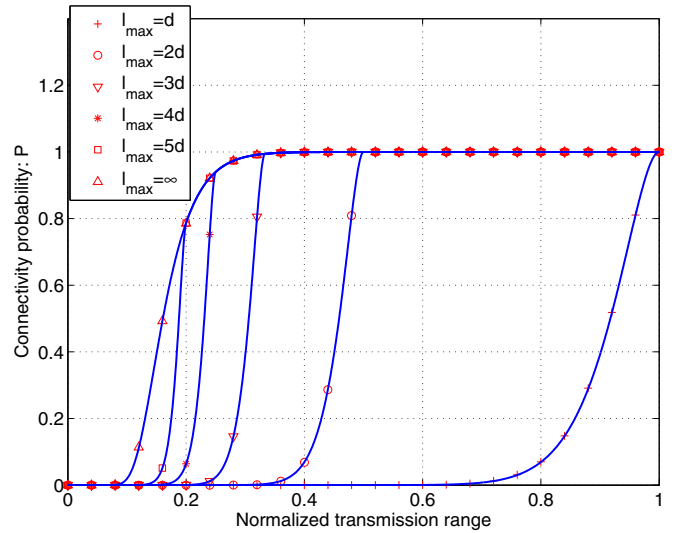
Fig. 2. Connectivity probabilities of networks with $l_{\max} = 2d$.

Fig. 3. Connectivity probabilities of network with dynamic node population ($\lambda = 20$) and various values of $l_{\max}$.

results indicate that increasing arrival rate or reducing node speed leads to a higher node density.

Fig. 2 shows the connectivity probabilities of networks under strict delay constraint. The delay constraint imposes a bound on maximum information propagation distance of $l_{\max} = 2d$, which translates to a maximum delay of $t_{\max} = 4t_p + 2\frac{d}{c}$. Perfect match is observed between the simulation results and the analytical results. It's interesting to note that when the normalized transmission range is small, $e.g.$, $\bar{d} < 0.5$, increasing node density leads to smaller connectivity probability. This seemingly contra-intuitive result is contributed by the limit on the maximum node distance, $i.e.$, the larger the number of nodes in a section of length L, the less likely that all the nodes will fall in a section with length $l_{\max} < L$ simultaneously. On the other hand, when $\bar{d} \geq 0.5$, we have $\bar{l}_{\max} = 2\bar{d} \geq 1$, which is equivalent to the case of no delay constraint. In addition, with the increase of n or λ, the transition from 0 connectivity to 100% connectivity requires only a small variation in $\bar{d}$. We denote the value of $\bar{d}$ corresponding to the $0 \to 100\%$ probability transition as critical transmission range, $\bar{d}_0(n)$, $i.e.$ with $n \to \infty$, if $\bar{d} < \bar{d}_0(n)$, then $P_n \to 0$; if $\bar{d} > \bar{d}_0(n)$, then $P_n \to 1$.

The impact of delay constraint (in the form of $l_{\max}$) on connectivity probability is investigated in Fig. 3 for network with dynamic node population and fixed value of $\lambda = 20$. The marks are obtained through empirical simulation, and the curves are calculated through theoretical results. It can be seen from the figure that the critical transmission range, which is the transition point from 0 connectivity to 100% connectivity, decreases monotonically as $l_{\max}$ increases. Small values of $l_{\max}$, or strict delay constraint, seriously limits the network connectivity. With the increase of $l_{\max}$, the performance of the delay constrained system gradually converges to the performance of systems without delay constraint.

VI. CONCLUSIONS

The connectivity of mobile linear networks with high speed mobile nodes, dynamic node populations, and strict delay constraint was investigated. With the tools from M/G/∞ queuing system, a new mobility model was developed to represent the steady state mobility properties that incorporate the effects of random node arrival, time-varying node speed, and distinct behaviors of nodes following different traffic patterns. The statistical properties of network connectivity were investigated with the help of the new mobility model and a novel geometry-assisted analytical method. The impacts of key network parameters, such as node arrival rate, time-varying node speed, and transmission delay constraint, are incorporated into exact closed-form expressions of connectivity probabilities. It is observed through numerical examples that the strict constraint on transmission delay seriously limits the connectivity probability of mobile networks.

APPENDIX

A. Proof of Lemma 3

The extended distance vector, $\tilde{\mathbf{Y}}$, can be expressed as a linear transformation of the ordered vector, $\mathbf{X}_{(o)}$. It can be easily shown that the Jacobian of the transformation is 1, and the result immediately follows.

B. Proof of Lemma 4

Let h denote the number of hops, then the one way transmission delay is $t_d = (h-1)t_p + l/c$. The number of hops involved during transmission over distance l can be bounded as $m + 1 \leq h \leq 2m + 1$, which can be proved by contradiction. If there are $h \leq m$ hops between the two nodes, then the maximum distance that is covered by the h hops is $hd \leq md < l$, which can not cover the distance between the two nodes. Thus $h \geq m + 1$. If there are $h \geq 2m + 2$ hops between the two nodes, then the total distance covered by the h hops can be written as

$$l = \sum_{k=1}^{2m+2} l_k + \sum_{k=2m+3}^{h} l_k \geq \sum_{k=1}^{m+1} \left(l_{2k-1} + l_{2k} \right), \qquad (19)$$

where l_k is the distance covered by the k-th hop. Since the distance covered by two consecutive hops must satisfy $l_{2k-1} + l_{2k} > d$, we have $l > (m+1)d$, which contradicts with $l \leq (m+1)d$. Thus $h \leq 2m + 1$.

Substituting the hop bound into the expression of t_d, and noting that $md < l \leq (m+1)d$, we have the result in (12).

C. Proof of Lemma 5

Denote $F(x) = \sum_{m=n+1}^{+\infty} \binom{m-1}{n} \frac{x^m}{m!}$. Differentiating $F(x)$ with respect to x yields

$$F'(x) = \frac{1}{n!} \sum_{m=n+1}^{+\infty} \frac{x^{m-1}}{(m-1-n)!} = \frac{x^n}{n!} e^x.$$

Performing integration over $F(x)$ leads to

$$F(x) = F(0) + \frac{1}{n!} \int_0^x t^n e^t dt = F(0) - \frac{(-1)^n}{n!} \gamma(n+1, -x).$$

Since $F(0) = 0$, the proof is complete.

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