Connectivity of Mobile Linear Networks with Dynamic Node Population and Delay Constraint

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***Abstract*—The connectivity properties of a mobile linear network with high speed mobile nodes and strict delay constraint are investigated. A new mobility model is developed to represent the steady state node distribution, and it accurately captures the statistical properties of random node arrival, time-varying node speed, and the distinct behaviors of nodes following different traffic patterns. With the mobility model, the statistical properties of network connectivity are studied and identified. Unlike most previous works that do not consider the impacts of transmission latency, which is critical for real time applications, this paper identifies the quantitative relationship between network connectivity and delay constraint. The results are applicable to both delay constrained networks and delay tolerant networks. The connectivity analysis is performed with a novel geometry-assisted analytical method. Exact connectivity probability expressions are developed by using the volumes of a hypercube intersected by a hyperplane, and a hyperpyramid. The geometry-assisted analytical method significantly simplifies the connectivity analysis.**

***Index Terms*—Network connectivity, mobile linear network, delay constraint, n-cube.**

I. Introduction

NETWORK connectivity is a critical metric for the planning, design, and evaluation of ad hoc networks. Two nodes in a network are connected if they can exchange information with each other, either directly or indirectly, within a certain latency constraint. A network is said to be connected if any pair of nodes in it are connected.

In many practical networks, such as vehicular ad hoc network (VANET) [1], [2], the nodes are moving at a high speed, and this results in a rapidly changing network topology with dynamic node population, which has profound impacts on network connectivity [3]. If the network is delay tolerant, then node mobility can improve network connectivity by utilizing a store-and-forward scheme, which allows intermediate nodes to temporarily store information and deliver it at a different location [4], [5]. On the other hand, most practical networks have strict constraints on transmission latency, which requires intermediate nodes to immediately forward received information to the next hop. Such a scheme is denoted as receive-and-forward in this paper. Nodes employing receive-and-forward need to maintain a relatively high transmission power to ensure the connectivity of the entire network. If the transmission power is too low, the nodes might be separated into isolated clusters; if the transmission power is too high, it will generate unnecessary interference beyond the intended receiver. Therefore, it's essential to identify the impacts of key network parameters, such as node mobility, delay constraint, transmission power (or transmission range), on the connectivity probability of a mobile linear network.

The study of network connectivity has attracted considerable interests recently [6] – [15]. Most of the connectivity analyses were performed for networks with randomly distributed stationary nodes [6] – [10]. In [6], the asymptotic critical transmission power of a disk-shaped two-dimension (2D) network is expressed as a scaling function of the number of nodes, n, when $n \to \infty$. A connectivity upper bound was derived in [7] for a special 2D network with a triangular lattice topology. VANET usually takes a one-dimension (1D) linear model with all the nodes distributed along a straight line. An approximated connectivity probability for a 1D network is presented in [8]. Exact connectivity probability of a 1D network with uniformly distributed stationary nodes is obtained in [9] and [10] with different approaches. Results in the above mentioned works are based on the assumption of stationary nodes. There are limited works on the connectivity of networks with mobile nodes [3], [5], [11] – [15]. The critical transmission range in a sparse mobile network is studied in [11] with computer simulations. Hybrid numerical-simulation analysis is performed in [12] to identity the approximated connectivity of a 2D network with stationary or mobile nodes. The exact connectivity probability of two nodes with distance l on a linear network modeled by a Poisson point process is presented in [13], and the result was extended in [14] by considering the effects of interference. A comprehensive mobility model for VANET is presented in [15] by considering the arrival and departure of nodes at predefined entry and exit points along a highway.

Most of the aforementioned works employ the receive-and-forward scheme, where the transmission delay is mainly contributed by the processing time of the forwarding operations performed at the intermediate nodes. As a result, the one-way transmission delay is directly related to the number of hops involved during the information delivery process. To meet the strict latency constraint of real time applications, it is necessary to limit the maximum number of hops involved during information transmission, and such a limit has significant impacts on network connectivity. To the best of the author's knowledge, there is no works in the literature devoted to the identification of mobile network connectivity under delay constraint.

This paper focuses on the analysis of network connectivity for a linear network with high speed mobile nodes, dynamic node population, and strict delay constraint. A new mobility model is developed with the tools and theories from M/G/∞ queuing systems [16], and it captures the effects of random node arrival, time-varying node speed, and distinct behaviors

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of nodes following different traffic patterns. The mobility model is more comprehensive and practical compared to the stationary network models used by most previous works. With the new mobility model, the statistical properties of network connectivity are studied. The impacts of delay constraint on network connectivity are investigated with an analytical bound that establishes the quantitative relationship among transmission delay, source-destination distance, and the number of hops required for transmission.

The connectivity analysis is performed with the assistance of an innovative geometry-assisted analytical method. Specifically, the volumes of two n-dimensional convex polytopes, including a hypercube (n-cube) intersected by a hyperplane, and a hyperpyramid, are derived. The geometric results significantly simplify the connectivity analysis, and they lead to exact closed-form network connectivity probability expressions. The results provide insights on the operations for both delay constrained network with the receive-and-forward scheme, and delay tolerant network with the store-and-forward scheme. Numerical examples demonstrate that the new analytical results can accurately capture the statistical properties of mobile linear networks with fast changing network topologies and node population, and they can be used to guide the design and analysis of VANET and other linear networks.

The remainder of this paper is organized as follows. Section II presents a new mobility model of a mobile linear network. Section III is devoted to the geometric analysis of the volumes of a hyperplane-intersected n-cube, and a hyperpyramid. Section IV presents the exact connectivity probability for a mobile linear network with the new geometry-assisted analytical method. Numerical examples are presented in Section V, and Section VI concludes the paper.

II. System Model and Preliminary Statistics

A. Mobility Model

Consider a section of a multi-lane unidirectional highway defined by the interval $\mathcal{L} = [0, L]$. Each node enters the network at $x = 0$, and exits at $x = L$. The node mobility is modeled after the following assumptions.

A.1) Nodes are divided into I classes corresponding to vehicles on different lanes. Nodes belonging to the same class share independently and identically distributed (i.i.d.) mobility properties.

A.2) A class i node enters the network following a Poisson distribution with arrival rate λ_{0i}, $i = 1, \cdots, I$.

A.3) The time that a class i node spends on a section of the highway, $[x_0, x_0 + x]$, is a random variable (RV), $T_i(x)$, with mean proportional to the section length x, *i.e.*,

$$\mu_{Ti(x)} = \int_0^\infty \tau f_{T_i(x)}(\tau) d\tau = \frac{x}{\nu_i}, \tag{1}$$

where $f_{T_i(x)}(\tau)$ is the probability density function (pdf) of $T_i(x)$, and ν_i is a scaling factor related to the distribution of class-i node speed.

A.4) Nodes can freely pass each other.

Assumption A.3) can be satisfied by assuming that the node speed is a stationary random process. In this case, $T_i(x)$ can be written as $T_i(x) = \int_{x_0}^{x_0+x} \frac{1}{V_i(y)} dy$, where $V_i(y)$ is the random speed for a class-i node at location y. Taking expectation on both sides of the above equation leads to $\mu_{T_i(x)} = \mathbb{E}\left[\frac{1}{V_i(y)}\right] x$, or $\nu_i = 1/\mathbb{E}\left[\frac{1}{V_i(y)}\right]$, with $\mathbb{E}(\cdot)$ denoting mathematical expectation. The parameter ν_i is independent of location or time due to the stationary assumption.

With the above mobility assumptions, at any moment t, the number of nodes inside $[0, L]$, $N(t)$, and the location of a given node inside $[0, L]$, $X(t)$, are random variables. We are interested in the steady state distribution of $N(t)$ and $X(t)$ as $t \to \infty$, and the results are summarized as follows.

Lemma 1: Consider a length-L linear network with node mobility described in assumptions A.1) - A.4). At steady state ($t \to \infty$), the number of nodes in $[0, L]$ follows a Poisson distribution with parameter $\lambda = L \sum_{i=1}^{I} \frac{\lambda_{0i}}{\nu_i}$, and these nodes are independently and uniformly distributed inside $[0, L]$.

Proof: 1) Based on assumptions A.2) and A.3), each class of nodes can be modeled as an $M/G/\infty$ queuing system [16], given the facts that the nodes interarrival time is exponentially distributed with Markovian property (M), the service time, $T_i(L)$, is generally distributed (G), and all nodes can be served immediately upon their arrival (∞ number of servers). Let $N_i(t)$ denote the number of class-i nodes inside $[0, L]$ at time t, then mapping the analysis of $M/G/\infty$ queue, we have

$$P\{N_i(t)=n\}=\sum_{k=n}^{\infty} P\{N_i(t)=n|K_i(t)=k\}\, P\{K_i(t)=k\}\,, \tag{2}$$

where $K_i(t)$ is the number of class-i nodes that arrive during $[0, t]$, and it follows a Poisson distribution with parameter $\lambda_{0i}t$.

Since a node is either inside or outside $[0, L]$, $N_i(t)$ conditioned on $K_i(t)$ follows a binomial distribution with parameter $\alpha_i(t)$, which is defined as the probability that a node arriving between $[0, t]$ is still within $[0, L]$ at time t. For a node arriving between $[0, t]$, $\alpha_i(t) = \int_0^t P\{T_i(L) \geq t - \tau\} f_{t_0|t}(\tau) d\tau$, where $T_i(L)$ is the amount of time that a class-i node spends on $[0, L]$, and $f_{t_0|t}(\tau)$ is the conditional pdf of the node's arrival time, t_0, given the fact that the node arrived between $[0, t]$. For Poisson arrival, $f_{t_0|t}(\tau) = \frac{1}{t}$, for $0 \leq \tau \leq t$ [17, Theorem 5.2]. Let $F_{T_i(L)}(\tau)$ denote the cumulative distribution function (cdf) of $T_i(L)$, we have

$$\alpha_i(t) = \frac{1}{t} \int_0^t \left[1 - F_{T_i(L)}(\tau)\right] d\tau. \tag{3}$$

Substituting the binomial distribution, $P(N_i(t) = n|K_i(t) = k)$, into (2) leads to $P\{N_i(t) = n\} = \frac{\lambda_i^n(t)}{n!} e^{-\lambda_i(t)}$, which is a Poisson distribution with parameter $\lambda_i(t) = \lambda_{0i} \int_0^t \left[1 - F_{T_i(L)}(\tau)\right] d\tau$. Thus $N_i \triangleq \lim_{t\to\infty} N_i(t)$ is Poisson distributed with parameter $\lambda_i \triangleq \lim_{t\to\infty} \lambda_i(t) = \frac{\lambda_{0i} L}{\nu_i}$. Since the sum of independent Poisson RVs is still Poisson distributed [17], the total number of nodes inside $[0, L]$, $N = \sum_{i=1}^{I} N_i$, is a Poisson RV with parameter $\lambda = L \sum_{i=1}^{I} \frac{\lambda_{0i}}{\nu_i}$.

2) Let $X_i(t)$ denote the location of a class-i node at time t, then

$$P\{X_i(t) < x\} = \frac{1}{t} \int_0^t \left[1 - F_{T_i(x)}(t - \tau)\right] d\tau. \tag{4}$$

Let X_i denote the location of a class-i node inside $[0, L]$ as $t \to \infty$, then for any interval

$[a, b] \subseteq [0, L]$, the probability $P\{a \leq X_i \leq b\} = \lim_{t\to\infty} P\{a \leq X_i(t) \leq b | 0 \leq X_i(t) \leq L\}$ can be written as

$$P\{a \leq X_i \leq b\} = \frac{P\{X_i \leq b\} - P\{X_i \leq a\}}{P\{0 \leq X_i \leq L\}} = \frac{b-a}{L}, \quad (5)$$

where the second equality is obtained from (1) and (4). Since the above probability is true for all $[a, b] \subseteq [0, L]$, $X_i \sim U([0, L])$. ■

The result in Lemma 1 indicates that the steady state node distribution for the mobile linear network with assumptions A.1) - A.4) can be modeled by a homogeneous Poisson process [13], [18], with parameter $\lambda = L \sum_{i=1}^{I} \frac{\lambda_{0i}}{\nu_i}$.

B. Preliminary Statistics

Let X_m, for $m = 1, 2, \cdots, n$, denote the position of n independent nodes uniformly distributed over the interval $[0, L]$, *i.e.*, $X_m \sim U([0, L])$. The number of nodes, $N = n$, is a Poisson RV with parameter $\lambda = L \sum_{i=1}^{I} \frac{\lambda_{0i}}{\nu_i}$.

Ordering the n RVs in an ascending order yields a group of new RVs, $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. The joint distribution of the ordered RVs are given in the following Lemma [19].

Lemma 2: Define $\mathbf{X}_{(o)} = [X_{(1)}, X_{(2)}, \cdots, X_{(n)}]^T$, where $\mathbf{A}^T$ denotes matrix transpose, then the pdf of $\mathbf{X}_{(o)}$ can be written as $f_{\mathbf{X}_{(o)}}(x_1, x_2, \cdots, x_n) = \frac{n!}{L^n}$, for $0 \leq x_1 \leq \cdots \leq x_n \leq L$. ■

The study of node connectivity requires the investigation of the distribution of the distance between node pairs. Define the size $(n-1) \times 1$ distance vector as $\mathbf{Y} = [Y_1, Y_2, \cdots, Y_{n-1}]^T$ with $Y_m = X_{(m+1)} - X_{(m)}$. To facilitate analysis, denote $Y_0 = X_{(1)}$. Prefixing $\mathbf{Y}$ with Y_0 leads to an extended distance vector, $\tilde{\mathbf{Y}} = [Y_0, Y_1, \cdots, Y_{n-1}]^T$. The distribution of $\tilde{\mathbf{Y}}$ is presented in the following Lemma.

Lemma 3: The pdf of the extended distance random vector, $\tilde{\mathbf{Y}}$, is $f_{\tilde{\mathbf{Y}}}(y_0, y_1, \cdots, y_{n-1}) = \frac{n!}{L^n}$, if $\sum_{m=0}^{n-1} y_m \leq L$ and $0 \leq y_m \leq L$, for $m = 0, \cdots, n-1$, and $f_{\tilde{\mathbf{Y}}}(y_0, y_1, \cdots, y_{n-1}) = 0$ otherwise.

Proof: The proof is in Appendix A. ■

The above preliminary statistics will be used to facilitate the connectivity analysis. The network connectivity will be investigated with a new geometry-assisted analytical method, which translates the derivation of connectivity probabilities to the evaluation of the volumes of certain convex polytopes.

III. Geometric Results

In this section, the geometric properties of two n-dimensional polytopes, $\mathcal{D}_n(d, L) = \{\mathbf{y}_n | \sum_{m=1}^{n} y_m \leq L, \quad 0 \leq y_m \leq d\}$, and $\mathcal{T}_n(d, L) = \{\mathbf{y}_n | \sum_{m=1}^{n} y_m \leq L, d < y_m \leq L\}$, $\forall d, L \in \mathcal{R}^+$, and $n \in \mathcal{N}$, are evaluated, where $\mathbf{y}_n = [y_1, \cdots, y_n]^T$, $\mathcal{R}^+$ is the set of positive real numbers, and $\mathcal{N}$ is the set of natural numbers. The volume of $\mathcal{D}_n(d, L)$ is defined as $\text{Vol}[\mathcal{D}_n(d, L)] = \int\cdots\int_{\mathbf{y}_n \in \mathcal{D}_n(d,L)} d\mathbf{y}_n$, and $\text{Vol}[\mathcal{T}_n(d, L)]$ is defined in a similar manner.

A. Volume of $\mathcal{D}_n(d, L)$

The set, $\mathcal{D}_n(d, L)$, can be geometrically interpreted as an n-dimensional hypercube with edge length d, $\mathcal{C}_n(d) = \{\mathbf{y}_n | 0 \leq y_m \leq d, m = 1, \cdots, n\}$, intersected by an n-dimensional hyperplane, $\mathcal{P}_n(L) = \{\mathbf{y}_n | \sum_{m=1}^{n} y_m = L\}$, as illustrated in Figs. 1 and 2, for $n = 2$ and 3, respectively.

The direct evaluation of $\text{Vol}[\mathcal{D}_n(d, L)]$ for arbitrary n is rather complicated. To gain insights, we start from the simple case with $n = 2$, and then deduce the volume with arbitrary n from this simple result. For $n = 2$, the volume of $D_2(d, L)$ corresponds to the shaded areas as shown in Fig. 1. The areas of the three cases can be calculated by using basic geometry, and the result is

$$\text{Vol}[\mathcal{D}_2(d, L)] = \begin{cases} d^2, & 0 \leq \bar{d} < \frac{1}{2}, \\ \frac{1}{2}\left[L^2 - 2(L-d)^2\right], & \frac{1}{2} \leq \bar{d} < 1, \\ \frac{1}{2}L^2, & 1 \leq \bar{d}, \end{cases} \quad (6)$$

where $\bar{d} = d/L$. When n is large, it would be difficult to evaluate the volume graphically. To gain further insights on the volume of the polytope, we note the following identities regarding d^2,

$$d^2 = \frac{1}{2}\left[L^2 - 2(L-d)^2 + (L-2d)^2\right]. \quad (7)$$

Observing the volume for $n = 2$ with the help of (7), and the graphical representation of $\mathcal{D}_3(d, L)$ in Fig. 2, we find that the volume of $\mathcal{D}_n(d, L)$ at one level of $\bar{d}$ can be obtained by adding or removing a certain number of pyramids (triangles) from the volume at the previous level of $\bar{d}$. It is thus postulated that the volume of $\mathcal{D}_n(d, l)$ for arbitrary value of n can also be obtained by successively adding or removing a certain number of hyperpyramids. This conjecture is stated and proved in the following Theorem.

Theorem 1: The volume of the polytope, $\mathcal{D}_n(d, L)$, is

$$\text{Vol}[\mathcal{D}_n(d, L)] = V_n^k(d, L) \triangleq \frac{1}{n!} \sum_{m=0}^{k} (-1)^m \binom{n}{m} (L - md)^n,$$
$$\text{if } \bar{d} \in \mathcal{L}_k(n), \quad \text{for } k = 0, 1, \cdots, n, \quad (8)$$

where $\bar{d} = d/L$, and $\mathcal{L}_k(n) = \left[\frac{1}{k+1}, \frac{1}{k}\right)$ for $k = 1, \cdots, n-1$, $\mathcal{L}_0(n) = [1, \infty)$, and $\mathcal{L}_n(n) = \left[0, \frac{1}{n}\right)$.

Proof: The theorem is proved with mathematical induction and details are presented in Appendix B. ■

An interesting byproduct of Theorem 1 is a series expansion of d^n presented as follows.

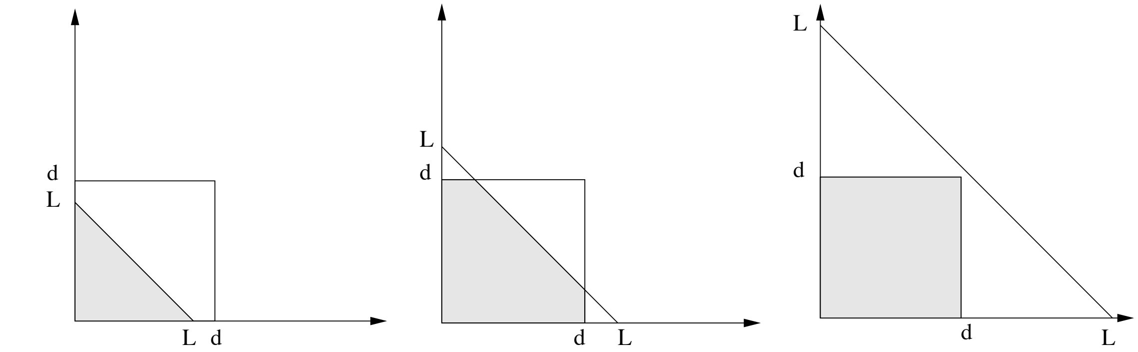
Corollary 1: For $n \in \mathcal{N}$ and $d, L \in \mathcal{R}^+$, d^n can be expressed by the following series expansion

$$d^n = \frac{1}{n!} \sum_{m=0}^{n} (-1)^m \binom{n}{m} (L - md)^n, \quad \forall L \geq nd. \quad (9)$$

Proof: In the case $L \geq nd$, the polytope, $\mathcal{D}_n(d, L)$ is the same as the n-cube $\mathcal{C}_n(d)$. Thus $\text{Vol}[\mathcal{D}_n(d, L)] = \text{Vol}[\mathcal{C}_n(d)] = d^n$, for $0 \leq \bar{d} \leq \frac{1}{n}$. Eqn. (9) immediately follows from Theorem 1. ■

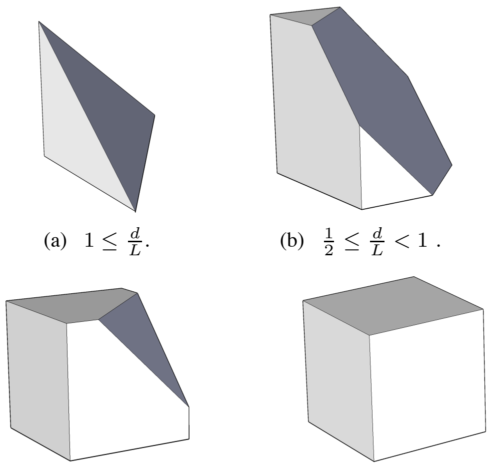
B. Volume of $\mathcal{T}_n(d, L)$

The set $\mathcal{T}_n(d, L)$, with $L > (n-1)d$, can be geometrically interpreted as an n-dimensional hyperpyramid with mutually orthogonal side edges, and it has the same geometric shape as illustrated in Fig. 2(a) for $n = 3$. The volume of $\mathcal{T}_n(d, L)$ is presented in the following Lemma.



(a) $1 \leq \frac{d}{L}$. (b) $\frac{1}{2} \leq \frac{d}{L} < 1$. (c) $0 \leq \frac{d}{L} < \frac{1}{2}$.

Fig. 1. Graphical representation of $\mathcal{D}_2(d, L)$.



(a) $1 \leq \frac{d}{L}$. (b) $\frac{1}{2} \leq \frac{d}{L} < 1$. (c) $\frac{1}{3} \leq \frac{d}{L} < \frac{1}{2}$. (d) $0 \leq \frac{d}{L} < \frac{1}{3}$.

Fig. 2. Graphical representation of $\mathcal{D}_3(d, L)$.

Proposition 1: The volume of the hyperpyramid, $\mathcal{T}_n(d, L)$, is

$$\text{Vol}\left[\mathcal{T}_n(d, L)\right] = \begin{cases} \frac{1}{n!}\left[L - nd\right]^n, & L \geq nd, \\ 0, & L < nd. \end{cases} \tag{10}$$

Proof: The proof is in Appendix C. ■

IV. Connectivity of Mobile Linear Networks

In this section, the connectivity properties of a delay constrained linear network are investigated with the help of the geometric results from the previous Section.

Before moving on to the connectivity analysis, we first establish an analytical bound that identifies the quantitative relationship between delay constraint and node distance for networks employing the receive-and-forward scheme. The result is presented in the following Lemma.

Lemma 4: Consider a source node and destination node separated by a distance $md < l \leq (m+1)d$, with d being the transmission range of a node. If the next hop is chosen as the node that is the furthest one within the transmission range of the current node, then the one way transmission delay between the source-destination node pair is bounded by

$$l\left(\frac{t_p}{d} + \frac{1}{c}\right) - t_p \leq t_d < l\left(2\frac{t_p}{d} + \frac{1}{c}\right), \tag{11}$$

where t_p is the processing time at one intermediate node, and c is the speed of light.

Proof: The proof is in Appendix D ■

The result in Lemma 4 indicates that the transmission delay is proportional to the source-destination distance. Thus a certain delay constraint can be achieved by limiting the source-destination distance. Given delay constraint $t_{\max}$, two nodes with distance l satisfying $l \leq l_{\max} \triangleq t_{\max} / \left[\left(2\frac{t_p}{d} + \frac{1}{c}\right)\right]$ is guaranteed to have $t_d < t_{\max}$. In the following analysis, we will study the impacts of delay constraint $t_{\max}$ by limiting the maximum distance between node pairs with $l_{\max}$, which guarantees the maximum delay is less than $t_{\max}$.

A. Connectivity of an n-Node Network

We first study the connectivity of a linear network with a fixed number of n nodes, and the results will be used to assist the analysis of networks with dynamic number of nodes.

Theorem 2: For a linear network with n nodes uniformly distributed over a section with length L, if the maximum transmission range of each node is d, and the maximum distance between any two nodes is bounded by $l_{\max}(\geq d)$ due to a delay constraint, then the probability that all the n nodes are connected is

$$P_n(\bar{d}, \bar{l}_{\max}) = \begin{cases} \sum_{m=0}^{k}(-1)^m\binom{n-1}{m}\left[\left(\bar{l}_{\max} - m\bar{d}\right)^n + n(1-\bar{l}_{\max})(\bar{l}_{\max} - m\bar{d})^{n-1}\right], & \bar{d} \leq \bar{l}_{\max} \leq 1 \\ \sum_{m=0}^{j}(-1)^m\binom{n-1}{m}\left(1 - m\bar{d}\right)^n, & \bar{l}_{\max} > 1 \end{cases}$$

where $\bar{l}_{\max} = l_{\max}/L$, $\bar{d}/\bar{l}_{\max} \in \mathcal{L}_k(n-1)$, $\bar{d} \in \mathcal{L}_j(n-1)$, for $k, j = 1, \cdots, n-1$.

Proof: When $\bar{d} \leq \bar{l}_{\max} \leq 1$, the probability can be expressed as $P_n(\bar{d}, \bar{l}_{\max}) = P\left\{\mathbf{Y} \in \mathcal{D}_{n-1}(d, l_{\max})\right\}$, which can be alternatively written by

$$P_n(\bar{d}, \bar{l}_{\max}) = \underbrace{P\{Y_0 \in [0, L - l_{\max}), \mathbf{Y} \in \mathcal{D}_{n-1}(d, l_{\max})\}}_{P_1} + \underbrace{P\{Y_0 \in [L - l_{\max}, L - d), \mathbf{Y} \in \mathcal{D}_{n-1}(d, L - Y_0)\}}_{P_2} + \underbrace{P\{Y_0 \in [L - d, L]\}}_{P_3}.$$

The inclusion of Y_0 in the above expression enables the utilization of $\tilde{\mathbf{Y}}$, which has a constant valued pdf. A constant valued pdf allows the application of the volume result for the connectivity analysis.

The probability P_1 can be expressed as $P_1 = \frac{n!}{L^n}(L - l_{\max})\text{Vol}\left[\mathcal{D}_{n-1}(d, l_{\max})\right]$.

Similarly, the probability P_2 can be expressed as

$$P_2 = \frac{n!}{L^n}\int_{L-l_{\max}}^{L-d} \text{Vol}\left[\mathcal{D}_{n-1}(d, L-y_0)\right] dy_0. \quad (12)$$

Since $\text{Vol}\left[\mathcal{D}_{n-1}(d, L-l_{\max})\right]$ assumes different expressions when $\frac{d}{L-y_0}$ falls in different definition intervals, the integration limit of (12) needs to be partitioned into several sections as $[L-l_{\max}, L-d] = [L-l_{\max}, L-kd) \cup \left(\cup_{m=1}^{k-1}[L-(m+1)d, L-md)\right)$. With such a partition, the probability in (12) can be written as

$$P_2 = \frac{n!}{L^n}\left[\int_{L-l_{\max}}^{L-kd} V_{n-1}^k(d, L-y_0)dy_0 + \sum_{m=1}^{k-1}\int_{L-(m+1)d}^{L-md} V_{n-1}^m(d, L-y_0)dy_0\right], \frac{\bar{d}}{\bar{l}_{\max}} \in \mathcal{L}_k(n-1).$$

Solving the two integrals and simplifying lead to

$$P_2 = \sum_{m=0}^{k}(-1)^m\binom{n-1}{m}\left(\bar{l}_{\max} - m\bar{d}\right)^n - \bar{d}^n, \frac{\bar{d}}{\bar{l}_{\max}} \in \mathcal{L}_k(n-1).$$

The probability, P_3, can be calculated as $P_3 = \int_{L-d}^{L} f_{Y_0}(y_0)dy_0 = \bar{d}^n$, where $f_{Y_0}(y_0) = \frac{n}{L^n}(L-y_0)^{n-1}$ [17] is used in the above equation. Combining P_1, P_2 and P_3 in the above equations leads to the first equality in (12).

When $\bar{l}_{\max} > 1$, the probability can be expressed as $P_n(\bar{d}, \bar{l}_{\max}) = P\{\mathbf{Y} \in \mathcal{D}_{n-1}(d, L)\} = P_n(\bar{d}, \bar{l}_{\max} = 1)$, which leads to the second equality in (12). ■

The result in Theorem 2 gives the probability that all the nodes are connected. In case some node pairs in the network do not need to exchange information, the probability can be considered as a lower bound. The result presented in Theorem 2 can also be considered as the connectivity probability of a stationary network with n nodes uniformly distributed over a linear section of the network.

In addition to the case that all the nodes are connected with strict delay constraint, it is also of interests of the probability that at least one pair of nodes can communicate with each other. Such a probability gives a good indicator of the connectivity of a delay tolerant network with the store-and-forward scheme, because information can be eventually delivered to its destination with such a scheme as long as the network is not completely isolated.

Corollary 2: For the linear network described in Theorem 2, the probability that at least one pair of nodes are connected is

$$Q_n(\bar{d}) = \begin{cases} 1 - \left[1-(n-1)\bar{d}\right]^n, & 1 \geq (n-1)\bar{d}, \\ 1, & 1 < (n-1)\bar{d}. \end{cases} \quad (13)$$

Proof: The proof is in Appendix E. ■

B. Connectivity of Networks with Dynamic Node Population

Before proceeding to the connectivity probability of a mobile network with random number of nodes, we have the following Lemma that will be used during the connectivity analysis

Lemma 5:

$$\sum_{m=n+1}^{+\infty}\binom{m-1}{n}\frac{x^m}{m!} = -(-1)^n\frac{1}{n!}\gamma(n+1, -x), \quad (14)$$

where $\gamma(n,x) = \int_0^x t^{n-1}e^{-t}dt$ is the lower incomplete Gamma function.

Proof: The proof is in Appendix F. ■

Theorem 3: Consider a linear network of length L. The number of nodes in the network follows a Poisson distribution with parameter λ, and all the nodes are uniformly distributed over $[0, L]$. If the maximum transmission range of each node is d, and the maximum distance between any two nodes is bounded by $l_{\max}$, then the probability that all the nodes in the network are connected is shown in (15) at the top of the next page. In (15), $\bar{d}/\bar{l}_{\max} \in \mathcal{L}_k(\infty)$, $\bar{d} \in \mathcal{L}_j(\infty)$, $k, j = 1, \cdots, \infty$, and $\mathcal{L}_k(\infty) = \left[\frac{1}{k+1}, \frac{1}{k}\right)$.

Proof: The proof is in Appendix G. ■

We next evaluate the probability that at least one pair of nodes are connected.

Corollary 3: Consider the linear network described in Theorem 3, the probability that at least one pair of nodes are connected is

$$Q_\lambda(\bar{d}) = 1 - e^{-\lambda}\sum_{m=0}^{k+1}\frac{\lambda^m}{m!}\left[1-(n-1)\bar{d}\right]^m, \quad (16)$$
$$\text{for } \bar{d} \in \mathcal{L}_k(\infty), \quad k = 1, \cdots, \infty.$$

Proof: The proof is presented in Appendix H. ■

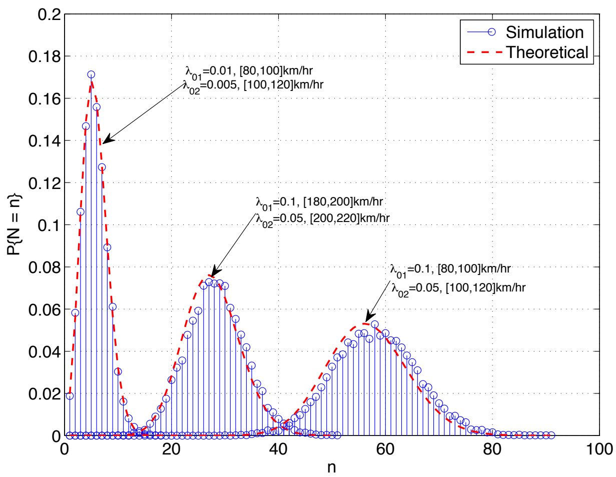
The results in Theorem 3 and Corollary 3 are the steady state connectivity probabilities of linear network with high speed mobile nodes and dynamic node population. They can be interpreted as the percentages of time that the network is fully connected, or not isolated, respectively.

V. Numerical Examples

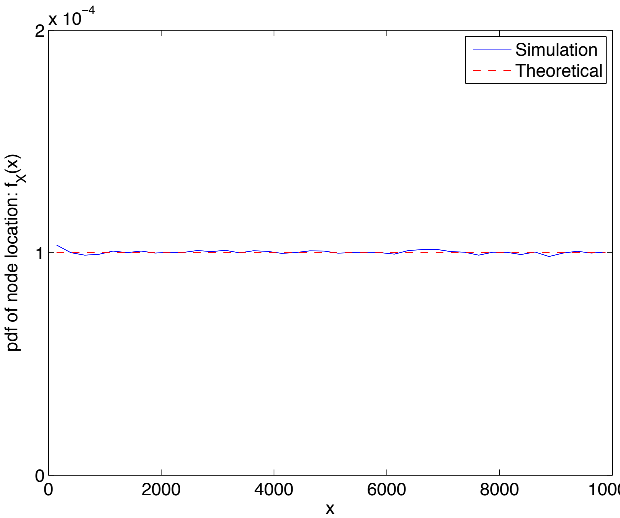
We first investigate the accuracy of the mobility model. Fig. 3 shows the steady state distribution of node population and node location under various configurations. In this example, the random nodes are divided into two classes. The speed of a class-i node is modeled as a stationary random process uniformly distributed in $[a_i, b_i]$. It can be easily shown that this implementation satisfies Assumption A.3) with parameter $\nu_i = \frac{b_i - a_i}{\log_e b_i - \log_e a_i}$. The parameters, λ_{0i} and $[a_i, b_i]$, of each class are shown in the figure. The length of the section is 10 km, and the data is collected over a period of 10,000 seconds. The analytical curves are generated by using the mobility model presented in Lemma 1. Comparison between the simulation results and analytical results reveals that the mobility model renders an accurate representation of the steady state distribution of random node population and node location. In addition, the results indicate that increasing arrival rate or reducing node speed leads to higher node density.

Fig. 4 compares the connectivity probabilities of networks with fixed node population and dynamic node population under different node densities. No delay constraint is assumed in this example. The results in the figure lead to the following observations. First, when the connectivity probability is large, network with fixed number of nodes always outperform network with dynamic number of nodes. Second, with the increase of n or λ, the transition from 0 connectivity to 100% connectivity requires only a small variation in $\bar{d}$. We denote the value of $\bar{d}$ corresponding to the $0 \rightarrow 100\%$ probability transition as critical transmission range, $\bar{d}_0(n)$, *i.e.*

$$P_\lambda(\bar{d},\bar{l}_{\max})=\begin{cases} e^{-\lambda}-\sum_{n=0}^{k}\frac{e^{-\lambda}}{n!}\left[\gamma(n+1,\bar{d}n\lambda-\bar{l}_{\max}\lambda)-(1-\bar{l}_{\max})\lambda^{n+1}\left(n\bar{d}-\bar{l}_{\max}\right)^{n}e^{\bar{l}_{\max}\lambda-\bar{d}n\lambda}\right], & \bar{l}_{\max}\le 1,\\ e^{-\lambda}-\sum_{n=0}^{j}\frac{e^{-\lambda}}{n!}\gamma(n+1,\bar{d}n\lambda-\lambda), & \bar{l}_{\max}>1, \end{cases} \tag{15}$$



(a) Distribution of number of nodes



(b) Distribution of node location

Fig. 3. Verification of mobility model as $t \to \infty$.

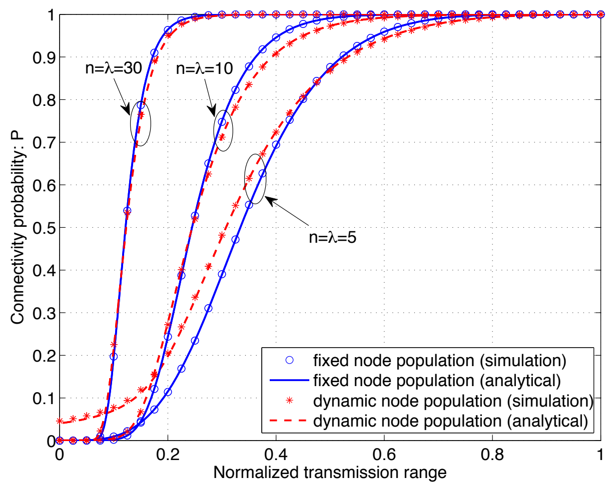


Fig. 4. Connectivity probabilities of networks with $l_{\max} = L$.

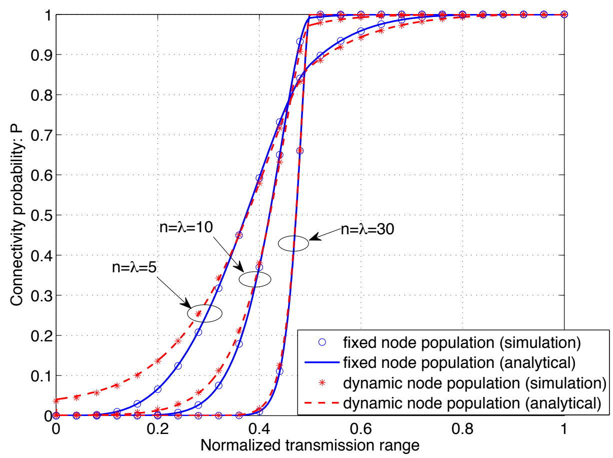


Fig. 5. Connectivity probabilities of networks with $l_{\max} = 2d$.

with $n \to \infty$, if $\bar{d} < \bar{d}_0(n)$, then $P_n \to 0$; if $\bar{d} > \bar{d}_0(n)$, then $P_n \to 1$. The critical transmission range decreases with the increase of n. Third, the connectivity probability at $\lambda = 5$ and $\bar{d} = 0$ is greater than 0. This is due to the fact that the network is defined as connected when $n = 0$ or 1, and these two cases contribute to the non-zero probability at $\bar{d} = 0$.

Fig. 5 shows the connectivity probabilities under strict delay constraint. The network has a bound on maximum node distance of $l_{\max} = 2d$, which translates to a maximum delay of $t_{\max} = 4t_p + 2\frac{d}{c}$. Other than the observations pointed out in Fig. 4, it's interesting to note that when the normalized transmission range is small, *e.g.*, $\bar{d} < 0.5$, increasing node density leads to smaller connectivity probability. This seemingly contra-intuitive result is contributed by the limit on the maximum node distance, *i.e.*, the larger the number of nodes in a section of length L, the less likely that all the nodes will fall in a section with length $l_{\max} < L$ simultaneously. On the other hand, when $\bar{d} \geq 0.5$, we have $\bar{l}_{\max} = 2\bar{d} \geq 1$, which is equivalent to the case of no delay constraint, and the results are the same as Fig. 4.

Fig. 6 investigates the probability that at least one pair of nodes are connected. This probability is used as an indicator of the connectivity probability of delay tolerant network with the store-and-forward scheme. The probability quickly saturates to 1 with a slight increase of transmission range. At the same node density, the connectivity probability of networks with fixed number of nodes increases faster compared to that of networks with dynamic number of nodes. The difference between the two networks gradually diminishes with the increase of node density. Comparing results in Fig. 4 and Fig. 6 indicates that the store-and-forward scheme outperforms receive-and-forward scheme in terms of connectivity probability, at the cost of unbounded transmission delay.

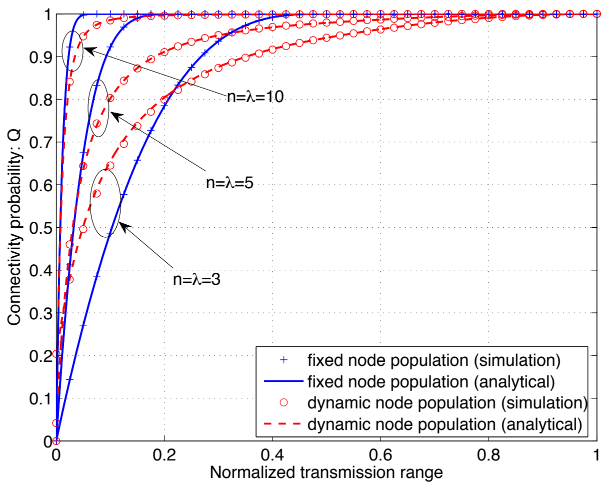


Fig. 6. Probability that at least one pair of nodes are connected.

VI. Conclusions

The connectivity of mobile linear networks with high speed mobile nodes, dynamic node populations, and strict delay constraint was investigated. With the tools from M/G/∞ queuing system, a new mobility model was developed to represent the steady state mobility properties that incorporate the effects of random node arrival, time-varying node speed, and distinct behaviors of nodes following different traffic patterns. The statistical properties of network connectivity were investigated with the new mobility model and a novel geometry-assisted analytical method. The impacts of key network parameters, such as node arrival rate, time-varying node speed, and transmission delay constraint, are incorporated into exact closed-form expressions of connectivity probabilities. It is observed that the strict constraint on transmission delay seriously limits the connectivity probability of mobile networks. The results in this paper are also applicable to bi-directional network, and they can be used to guide the planning, design, and evaluation of VANET and other mobile linear networks.

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Appendix

A. Proof of Lemma 3

The extended distance vector, $\tilde{\mathbf{Y}} = [Y_0, Y_1, \cdots, Y_{n-1}]^T$, can be expressed as a linear transformation of the ordered vector, $\mathbf{X}_{(o)}$. It can be easily shown that the Jacobian of the transformation is 1. Thus, $f_{\tilde{\mathbf{Y}}}(\mathbf{y}) = f_{\mathbf{X}_{(o)}}(\mathbf{x})$, and Lemma 3 immediately follows from Lemma 2.

B. Proof of Theorem 1

Proof by induction. The proof for $n = 1$ is trivial. Assume (8) is true for Vol $[\mathcal{D}_{n-1}(d, L)]$. The induction part of the proof is divided into three cases, $\bar{d} \in \mathcal{L}_k(n)$, for $k = 1, \cdots, n-1$, $\bar{d} \in \mathcal{L}_0(n)$, and $\bar{d} \in \mathcal{L}_n(n)$.

1) $\bar{d} \in \mathcal{L}_k(n)$, for $k = 1, \cdots, n-1$. Based on the volume definition, when $\bar{d} \in \mathcal{L}_k(n)$, we have

$$V_n^k(d, L) = \int_0^d \text{Vol}\left[\mathcal{D}_{n-1}(d, L - y_n)\right] dy_n,$$
$$= \int_{kd}^{L} V_{n-1}^k(d, z)dz + \int_{L-d}^{kd} V_{n-1}^{k-1}(d, z)dz, \quad (17)$$

where the integration interval in the first equality is partitioned as $[0, d] = [0, L - kd] \cup [L - kd, d]$ based on the definition interval of $V_{n-1}^k(d, L)$.

Solving the two integrals in (17) with the definition of $V_n^k(d, L)$ and simplifying lead to

$$V_n^k(d, L) = \frac{1}{n!}\left\{L^n + \sum_{m=1}^{k}(-1)^m \binom{n}{m}(L - md)^n\right\}, \quad (18)$$

where the identity, $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$, is used in the simplification. Eqn. (18) simplifies to (8).

2) $\bar{d} \in \mathcal{L}_n(n)$. The volume can be directly written as

$$V_n^n(d, L) = \int_0^d V_{n-1}^{n-1}(d, L - y_n)dy_n, \quad (19)$$

due to the fact that $0 \leq \frac{d}{L-y_n} < \frac{1}{n-1}$ in the entire integration interval $y_n \in [0, d]$. Simplifying the above equation with the definition of $V_{n-1}^{n-1}(d, L)$ leads to (8).

3) $\bar{d} \in \mathcal{L}_0(n)$. The condition $1 \leq \bar{d}$ means $0 < y_n \leq \sum_{m=1}^n y_m \leq L \leq d$. Thus the integration limit of y_n is $[0, L]$. In this case, $V_n^0(d, L)$ can be calculated as

$$V_n^0(d, L) = \int_0^L V_{n-1}^0(d, L - y_n)dy_n. \quad (20)$$

which can be simplified to $V_n^0(d, L) = \frac{1}{n!}L^n$.

C. Proof of Proposition 1

When $L < nd$, $\mathcal{T}_n(d, L)$ is an empty set, thus with volume 0. The result when $L \geq (n-1)d$ is proved with induction. When $n = 1$, Vol $[\mathcal{T}_1(d, L)] = L - d$. Assume the result is true for $n - 1$. Vol $[\mathcal{T}_n(d, L)]$ can be recursively expressed as

$$\text{Vol}\left[\mathcal{T}_n(d, L)\right] = \int_d^{L-(n-1)d} \text{Vol}\left[\mathcal{T}_{n-1}(d, L - y_n)\right] dy_n. \quad (21)$$

Simplifying the above result leads to (10).

D. Proof of Lemma 4

Let h denote the number of hops. The one way transmission delay can be expressed as $t_d = (h-1)t_p + l/c$. The number of hops involved during transmission over distance l can be bounded as $m + 1 \leq h \leq 2m + 1$, which can be proved by contradiction. If there are $h \leq m$ hops between the two nodes, then the maximum distance that is covered by the h hops is $hd \leq md < l$, which can not cover the distance between the source-destination node pair. Thus $h \geq m + 1$. If there are $h \geq 2m + 2$ hops between the two nodes, then the total distance covered by the h hops can be written as

$$l = \sum_{k=1}^{2m+2} l_k + \sum_{k=2m+3}^{h} l_k \geq \sum_{k=1}^{m+1}\left(l_{2k-1} + l_{2k}\right), \quad (22)$$

where l_k is the distance covered by the k-th hop. Since the distance covered by two consecutive hops must satisfy $l_{2k-1} + l_{2k} > d$, we have $l > (m+1)d$, which contradicts with $l \leq (m+1)d$. Thus $h \leq 2m+1$.

Substituting the hop bound into the expression of t_d, and noting that $md < l \leq (m+1)d$, we have the result in (11).

E. Proof of Corollary 2

Define $Q_n^c(\bar{d}) = 1 - Q_n(\bar{d})$ as the probability that all the nodes are isolated. If $L < (n-1)d$, then there must be at least two nodes with distance less than d, otherwise the accumulated distance of the $(n-1)$ adjacent node pairs will be larger than L. Thus $Q_n^c(\bar{d}) = 0$ for $L < (n-1)d$.

When $L \geq (n-1)d$, we have $Q_n^c(\bar{d}) = P\left\{Y_0 \in [0, L-(n-1)d], \mathbf{Y} \in \mathcal{T}_{n-1}(d, L-y_0)\right\}$, which can be calculated as $Q_n^c(\bar{d}) = \frac{n!}{L^n}\int_0^{L-(n-1)d} \text{Vol}\left[\mathcal{T}_{n-1}(d, L-y_0)\right] dy_0$. Combining the result with Proposition 1 leads to (13).

F. Proof of Lemma 5

Denote $F(x) = \sum_{m=n+1}^{+\infty} \binom{m-1}{n} \frac{x^m}{m!}$. Differentiating $F(x)$ with respect to x yields

$$F'(x) = \frac{1}{n!}\sum_{m=n+1}^{+\infty} \frac{x^{m-1}}{(m-1-n)!} = \frac{x^n}{n!}e^x.$$

Performing integration over $F(x)$ leads to

$$F(x) = F(0) + \frac{1}{n!}\int_0^x t^n e^t dt = F(0) - \frac{(-1)^n}{n!}\gamma(n+1, -x).$$

Since $F(0) = 0$, the proof is complete.

G. Proof of Theorem 3

If $\frac{1}{k+1} \leq \frac{d}{l_{\max}} < \frac{1}{k}$, then the connectivity probability can be expressed as $P_\lambda(\bar{d}, \bar{l}_{\max}) = \sum_{m=0}^{\infty} P_m(\bar{d}, \bar{l}_{\max}) P\{N = m\}$, which can be further written as

$$P_\lambda(\bar{d}, \bar{l}_{\max}) = e^{-\lambda} + e^{-\lambda}\sum_{m=1}^{\infty}\sum_{n=0}^{\min(m-1,k)} (-1)^n \binom{m-1}{n} \times \left[\frac{\beta_n^m}{m!} + (1-\bar{l}_{\max})\lambda\frac{\beta_n^{m-1}}{(m-1)!}\right], \quad (23)$$

where $\beta_n = \left(\bar{l}_{\max} - n\bar{d}\right)\lambda$, and $P_0(\bar{d}) = 1$ is used in the second equality. Exchanging the order of summation in (23), and noting the fact that $n \leq k < m$, we can obtain (15) by using the result from Lemma 5.

H. Proof of Corollary 3

If $\frac{1}{k+1} \leq \frac{d}{L} < \frac{1}{k}$, then the probability, $Q_\lambda(\bar{d}) = \sum_{m=0}^{\infty} Q_m\left(\bar{d}\right) P\{N = m\}$, can be expressed as

$$Q_\lambda(\bar{d}) = 1 + e^{-\lambda} + \sum_{m=2}^{k+1} Q_m\left(\bar{d}\right) P\{N = m\} + \sum_{m=k+2}^{\infty} P\{N = m\}.$$

Where $Q_0(\bar{d}) = Q_1(\bar{d}) = 1$ is used. Simplifying the above equation with Corollary 2 leads to (16).

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