Connectivity Analysis of a Mobile Vehicular Ad Hoc Network with Dynamic Node Population

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Abstract— **Network connectivity is a critical metric for the planning, design, and evaluation of ad hoc networks. In this paper, the connectivity properties of a linear vehicular ad hoc network (VANET) with high speed mobile nodes and dynamic node populations are investigated. The nodes are assumed to be arriving at the network following a Poisson distribution, and the speed of each node is modeled as a wide sense stationary ergodic random process. Based on these assumptions, a new mobility model is developed to represent the steady state node distributions in terms of random node locations and random node populations. The mobility model accurately captures the statistical properties of a mobile network with rapidly changing network topologies and network densities. Exact closed-form connectivity probability expressions are developed with the assistance of the volume of an n-dimensional hypercube intersected by an n-dimensional hyperplane. The impacts of key network parameters, such as node arrival rate, random node speed, and node transmission range, are incorporated in the analytical expressions. The results can be used to determine the transmission power that can ensure network connectivity while minimizing interference in a VANET.**

keywords : VANET, network connectivity, network mobility, outage transmission range.

I. Introduction

Recently there have been significant efforts from government, industry, and academia devoted to the development of vehicle-to-vehicle and vehicle-to-infrastructure communication networks [1]. The integration of communication technologies with transportation systems leads to a plethora of applications, such as collision avoidance, traffic congestion control, intelligent cruise control, emergency warning, mobile business, and mobile entertainment, etc. These applications will improve vehicle safety, increase highway throughput, reduce vehicle related pollutions, and increase passenger comforts.

Vehicles equipped with communication facilities form a vehicular ad hoc network (VANET) [1], [2]. Information in VANET can be distributed by allowing nodes (vehicles) in a network to relay each other's information from source to destination, thus the proper operation of a VANET relies on the connectivity among nodes in the network [3]. One of the most distinguishing features of VANET is that all nodes are constantly moving at high speeds, and this results in a rapidly changing network topology with dynamic node numbers. To ensure the connectivity of the entire network, the nodes need to maintain a relatively high transmission power such that all the moving nodes can communicate with each other, either directly or indirectly, with high probability. If the transmission power is too low, the nodes might be separated into isolated clusters, and this will negatively impact the performance and practical usability of a VANET. On the other hand, if the transmission power of a node is too high, it will generate unnecessary interferences beyond its intended receiver. Therefore, it's essential to identify the impacts of key network parameters, such as node mobility, node density and distribution, transmission power (or transmission range), on the connectivity probability of a VANET.

The study of network connectivity has attracted considerable interests recently [4] – [18]. Most of the connectivity analysis was performed for networks with randomly distributed stationary nodes [4] – [8]. In [4], the asymptotic critical transmission power of a disk-shaped two-dimension (2D) network is expressed as a scaling function of the number of nodes, n, when $n \to \infty$. A connectivity upper bound was derived in [5] for a special 2D network with a triangular lattice topology. VANET usually takes a one dimension (1D) model with all the nodes distributed along a straight line. In [6], an approximated connectivity probability for a 1D network is presented as a function of network size and node transmission range. Exact connectivity probability of a 1D network with uniformly distributed stationary nodes is obtained in [7] and [8] with two different approaches. All of the aforementioned works are based on the assumption of stationary nodes, and they cannot be applied to mobile networks. Network mobility has profound impacts on the connectivity of practical networks [9]. There are limited works on the connectivity of network with mobile nodes [9] – [14]. A Poisson arrival location model (PALM) is introduced in [10] for a linear VANET, and it assumes that nodes with deterministic speed arrive at a network follows a Poisson distribution. The randomness in node speed is considered in [11] for VANET with sparse or dense node populations. The lower bound and upper bound of connectivity probability of a mobile VANET is presented in [9]. A comprehensive mobility model for VANET is presented in [13] by considering the arrival and departure of nodes at predefined entry and exit points along a highway. In [14], the connectivity of a delay tolerant mobile networks is studied by leveraging on the storage capability and movement of intermediate nodes.

This paper focuses on the identification of network connectivity for a linear VANET with high speed mobile nodes and dynamic node populations. The nodes are assumed to be arriving at the network following a Poisson distribution, and the speed of each node is modeled as a wide sense stationary (WSS) ergodic random process. Based on the statistical properties of node movement, a new mobility model is developed to represent the distributions of node population and node location when the network enters steady state. The model accurately captures the statistical properties of a mobile VANET with ever changing network topologies and node populations, and it's more comprehensive and practical compared to the stationary network models used by most previous works. With the help of the new mobility model, the statistical properties of network connectivity of mobile VANET are studied and identified. The connectivity analysis is performed with the assistance of the volume of an n-dimensional polytope obtained by intersecting an n-dimensional hypercube (n-cube) with an n-dimensional hyperplane. The results lead to exact closed-form network connectivity probability expressions, which are expressed as functions of key network parameters, including node arrival rate, node speed, node population distribution, node transmission range, and network length. A new performance measure, outage transmission range, is defined to quantitatively identify the relationship between node transmission range and network distributions.

The remainder of this paper is organized as follows. Section II presents a new mobility model of a mobile VANET. Section III is devoted to the study of the volume of an n-cube intersected by a hyperplane, and the result will be used to facilitate the connectivity analysis. Section IV presents the exact connectivity probability for a VANET with mobile nodes. The concept of outage transmission range is introduced in Section V, and approximated expressions of outage transmission range are presented in this section. Numerical examples are presented in Section VI, and Section VII concludes the paper.

II. Mobility Model

Consider a section of a unidirectional highway defined by the interval $\mathcal{L} = [0, L]$. Each node enters the highway at $x = 0$, and exits and $x = L$. The mobility of the nodes in the VANET are modeled after the following two assumptions.

A.1) The nodes enter the VANET following a Poisson distribution with arrival rate λ_0.

A.2) The node speed, $V(t)$, is a WSS ergodic random process with mean, μ_V.

For a given node entering the section at $t = t_0$, the location of the node at $t \geq t_0$ is

$$X(t, t_0) = \int_{t_0}^{t} V(\tau) d\tau. \tag{1}$$

The statistical properties of node location, $X(t, t_0)$, play a critical role on network connectivity. To simplify analysis, let's first consider a simple case with $V(t) = \mu_V$. The result obtained under this simple assumption will be extended to the general case of random $V(t)$ with the help of the ergodic property of $V(t)$ as $t \to \infty$. With $V(t) = \mu_V$, (1) can be simplified to

$$X(t, t_0) = (t - t_0)\mu_V. \tag{2}$$

Since nodes arrive at the network according to a Poisson distribution, the arrival time t_0 is a random variable (RV). Given the fact that there is a node arriving in $[0, t]$, the arrival time, t_0, is uniformly distributed on the interval $[0, t]$ [19, Theorem 5.2]. As a result, $X(t, t_0)$ is uniformly distributed in $[0, \mu_V t]$. We are only interested in those nodes that fall in the interval $[0, L]$. Let X denote the location of a node inside $[0, L]$. For any interval $[a, b] \subseteq [0, L]$, we have

$$\begin{aligned} P\{a \leq X \leq b\} &= P\{a \leq X(t, t_0) \leq b | 0 \leq X(t, t_0) \leq L\}, \\ &= \frac{(b-a)/(\mu_V t)}{L/(\mu_V t)} = \frac{b-a}{L}. \end{aligned} \tag{3}$$

Since the above probability is true for all $[a, b] \subseteq [0, L]$, X is uniformly distributed in the interval $[0, L]$, *i.e.*, $X \sim U([0, L])$.

The number of nodes inside $[0, L]$ at any time t is also a random variable. Let $N(t)$ denote the number of nodes in $[0, L]$ at time t, and $K(t)$ denote the total number of nodes that arrive at the time interval $[0, t]$, then the probability mass function (PMF) of $N(t)$ can be written as

$$P\{N(t) = n\} = \sum_{k=n}^{\infty} P\{N(t) = n | K(t) = k\} P\{K(t) = k\}. \tag{4}$$

With a node arrival rate of λ_0, we have

$$P\{K(t) = k\} = \frac{(\lambda_0 t)^k}{k!} e^{-\lambda_0 t}. \tag{5}$$

For any of the node arriving between $[0, t]$, the probability that it is still within $[0, L]$ at time t is $\alpha(t) \triangleq P\{0 \leq X(t, t_0) \leq L\} = L/(\mu_V t)$. Since a node is either inside or outside $[0, t]$, $N(t)$ conditioned on $K(t)$ follows a binomial distribution, and the probability $P\{N(t) = n | K(t) = k\}$ can thus be expressed as

$$P\{N(t) = n | K(t) = k\} = \binom{k}{n} [\alpha(t)]^n [1 - \alpha(t)]^{k-n}. \tag{6}$$

Substituting (5) and (6) into (4) yields

$$P\{N(t) = n\} = \frac{(L\lambda_0/\mu_V)^n}{n!} e^{-L\lambda_0/\mu_V}. \tag{7}$$

Thus, $N(t)$ is Poisson distributed with mean $\lambda = L\lambda_0/\mu_V$.

The above analysis is based on the assumption of constant node speed, $V(t) = \mu_V$. Next we will extend the results to system with speed being a random process as described in A.2). In this case, we are interested in the distribution of $X(t, t_0)$ as the system enters steady state, *i.e.*, $t \to \infty$. With the ergodic assumption of $V(t)$, given t_0, we have

$$\mu_V = \lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} V(\tau) d\tau. \tag{8}$$

Combining (1) and (8) leads to

$$\lim_{t \to \infty} X(t, t_0) = \lim_{t \to \infty} (t - t_0)\mu_V. \tag{9}$$

Comparing (2) with (9) reveals that the distribution of variable speed node asymptotically converges to the distribution of constant speed node as $t \to \infty$. Since the results in (3) and (7) are independent of t, they can be directly applied to system with random speed nodes as $t \to \infty$. The asymptotic behavior is also verified through simulation. The results are summarized in the following Lemma.

Lemma 1: Consider a length-L linear network with node mobility described in Assumptions A.1) and A.2). At steady state ($t \to \infty$), the number of nodes in $[0, L]$ follows a Poisson distribution with parameter $\lambda = L\lambda_0/\mu_V$, and these nodes are uniformly distributed inside $[0, L]$. ■

Based on the asymptotic mobility model, let X_m, for $m = 1, 2, \cdots, n$, denote the location of n independent nodes uniformly distributed over the interval $[0, L]$. The number of nodes, $N = n$, is a Poisson RV with parameter $\lambda = \frac{L\lambda_0}{\mu_V}$. Ordering the n RVs in ascending order yields a group of new RVs, $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$. Based on the results in order statistics, the joint distribution of $X_{(n)}$ is summarized as follows [20].

Lemma 2: Define the ordered vector of i.i.d. uniform RVs as $\mathbf{X}_{(o)} = [X_{(1)}, \cdots, X_{(n)}]$, then the probability density function (pdf) of $\mathbf{X}_{(o)}$ can be written as $f_{\mathbf{X}_{(o)}}(x_1, \cdots, x_n) = \frac{n!}{L^n}$, for $0 \le x_1 \le \cdots \le x_n \le L$. ■

The study of connectivity among nodes requires the investigation of the distribution of the distances among the node pairs. If the maximum transmission range of a node is d, then the probability that all the n nodes on the highway are connected is equivalent to the probability that the distance between any pair of adjacent nodes is no larger than d. Define a size $(n-1) \times 1$ distance vector as $\mathbf{Y} = [Y_1, Y_2, \cdots, Y_{n-1}]$, where $Y_m = X_{(m+1)} - X_{(m)}$ is the distance between the $(m+1)$-th and the m-th ordered nodes.

With the notations given above, the probability that n nodes in a VANET are connected can be calculated as

$$P_n(d_0) = \int \cdots \int_{\mathbf{y}_{n-1} \in \mathcal{D}_{n-1}(d,L)} f_{\mathbf{Y}}(y_1, \cdots, y_{n-1}) dy_1 \cdots dy_{n-1}, \quad (10)$$

where $f_{\mathbf{Y}}(y_1, \cdots, y_{n-1})$ is the pdf of the distance vector $\mathbf{Y}$, $d_0 = \frac{d}{L}$ with d being the node transmission range, $\mathbf{y}_{n-1} = [y_1, \cdots, y_{n-1}] \in \mathcal{L}^{1\times(n-1)}$, and $\mathcal{D}_{n-1}(d, L) \triangleq \left\{ \mathbf{y}_{n-1} \left| \sum_{m=1}^{n-1} y_m \le L, \quad 0 \le y_m \le d, m = 1, \cdots, n-1 \right. \right\}$. Since the node population $N = n$ is random, the average probability that a mobile VANET is connected can then be calculated as $P_\lambda(d_0) = \sum_{n=0}^{\infty} P_n(d_0) \frac{\lambda^n}{n!} e^{-\lambda}$.

The set, $\mathcal{D}_{n-1}(d, L)$, can be geometrically interpreted as an $(n-1)$-dimensional hypercube with edge length d, $\mathcal{C}_{n-1}(d) = \{\mathbf{y}_{n-1} \,| 0 \le y_m \le d, m = 1, \cdots, n-1\}$, intersected by an $(n-1)$-dimensional hyperplane, $\mathcal{P}_{n-1}(L) = \left\{ \mathbf{y}_{n-1} \left| \sum_{m=1}^{n-1} y_m = L \right. \right\}$. The result is an $(n-1)$-dimensional convex polytope. We are going to show during the analysis that the volume of $\mathcal{D}_{n-1}(d, L)$ is critical to the evaluation of the connectivity probability of the VANET. We present in the next section the volume of polytope $\mathcal{D}_n(d, L)$, and the results will be used to facilitate the network connectivity analysis.

III. Volume of the Polytope $\mathcal{D}_n(d, L)$

The volume of $\mathcal{D}_n(d, L)$, $\forall d, L \in \mathcal{R}^+$, and $n \in \mathcal{N}$, is evaluated in this section, where $\mathcal{R}^+$ is the set of positive real numbers, and $\mathcal{N}$ is the set of natural numbers. The volume of $\mathcal{D}_n(d, L)$ is defined as

$$\text{Vol}\left[\mathcal{D}_n(d, L)\right] = \int \cdots \int_{\mathbf{y}_n \in \mathcal{D}_n(d,L)} dy_1 \cdots dy_n. \quad (11)$$

Theorem 1: The volume of $\mathcal{D}_n(d, L)$ is

$$\text{Vol}\left[\mathcal{D}_n(d, L)\right] = V_n^k(d, L) \triangleq \frac{1}{n!} \sum_{m=0}^{k} (-1)^m \binom{n}{m} (L - md)^n,$$

$$\text{if} \quad \frac{d}{L} \in \mathcal{L}_k(n), \ \text{for} \ \ k = 1, \cdots, n, \quad (12)$$

where $\mathcal{L}_k(n) = \left[\frac{1}{k+1}, \frac{1}{k}\right)$ for $k = 0, 1, \cdots, n-1$, and $\mathcal{L}_k(n) = \left[0, \frac{1}{n}\right)$ for $k = n$.

Proof: Proof by induction. The proof for $n = 1$ is trivial. Assume (12) is true for $\text{Vol}\left[\mathcal{D}_{n-1}(d, L)\right]$. The induction part of the proof is divided into three cases, $\frac{d}{L} \in \mathcal{L}_0(n)$, $\frac{d}{L} \in \mathcal{L}_k(n)$, for $k = 1, \cdots, n-1$, and $\frac{d}{L} \in \mathcal{L}_n(n)$.

1) $\text{Vol}\left[\mathcal{D}_n(d, L)\right]$, $\frac{d}{L} \in \mathcal{L}_k(n)$, for $k = 1, \cdots, n-1$.

Based on the volume definition, when $d_0 \in \mathcal{L}_k(n)$, we have

$$V_n^k(d, L) = \int_0^d \text{Vol}\left[\mathcal{D}_{n-1}(d, L - y_n)\right] dy_n,$$
$$= \int_{kd}^{L} V_{n-1}^k(d, z) dz + \int_{L-d}^{kd} V_{n-1}^{k-1}(d, z) dz, \quad (13)$$

where the integration interval in the first equality is partitioned as $[0, d] = [0, L - kd] \cup [L - kd, d]$ based on the definition interval of $V_n^k(d, L)$.

Solving the two integrals in (13) with the definition of $V_n^k(d, L)$ and simplifying lead to

$$V_n^k(d, L) = \frac{1}{n!} \left\{ L^n + \sum_{m=1}^{k} (-1)^m \binom{n}{m} (L - md)^n \right\}, \quad (14)$$

where the identity

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}, \quad (15)$$

is used in the simplification. Eqn. (14) simplifies to (12).

2) $\text{Vol}\left[\mathcal{D}_n(d, L)\right]$, $\frac{d}{L} \in \mathcal{L}_n(n)$.

The volume at $d_0 \in \mathcal{L}_n(n)$, can be directly written as

$$V_n^n(d, L) = \int_0^d V_{n-1}^{n-1}(d, L - y_n) dy_n, \quad (16)$$

due to the fact that $0 \le \frac{d}{L - y_n} < \frac{1}{n-1}$ is in the entire integration interval $y_n \in [0, d]$. Simplifying the above equation with the definition of $V_{n-1}^{n-1}(d, L)$, and the combinatorics identities, $\binom{n-1}{0} = \binom{n}{0}$, and (15), lead to (12).

3) $\text{Vol}\left[\mathcal{D}_n(d, L)\right]$, $\frac{d}{L} \in \mathcal{L}_0(n)$.

The condition $1 \le \frac{d}{L}$ means $0 < y_n \le \sum_{m=1}^{n} y_m \le L \le d$. Thus the integration limit of y_n is $[0, L]$. Noting the fact that $1 \le \frac{d}{L - y_n}$, $\forall y_n \in [0, L]$, we have

$$V_n^0(d, L) = \int_0^L V_{n-1}^0(d, L - y_n) dy_n. \quad (17)$$

which can be simplified to $V_n^0(d, L) = \frac{1}{n!} L^n$.

Since the induction is true for $\text{Vol}\left[\mathcal{D}_n(d,L)\right]$ over all the definition intervals, the Theorem is proved. ■

An interesting byproduct of Theorem 1 is a series expansion of d^n, which is presented in the following Corollary.

Corollary 1: For $n \in \mathcal{N}$ and $d, L \in \mathcal{R}^+$, d^n can be expressed by the following series expansion

$$d^n = \frac{1}{n!}\sum_{m=0}^{n}(-1)^m\binom{n}{m}(L-md)^n, \quad \forall L \geq nd. \tag{18}$$

Proof: If $L \geq nd$, the polytope $\mathcal{D}_n(d,L)$ is the same as $\mathcal{C}_n(d)$, a hypercube with edge length d. Thus Eqn. (18) directly follows from $\text{vol}\left[\mathcal{C}_n(d)\right] = d^n$ and Theorem 1. ■

IV. Connectivity of Mobile VANET

In this section, the connectivity of a linear VANET with mobile nodes and dynamic node populations are investigated with the help of the volume of the polytope, $\mathcal{D}_n(d,L)$.

A. Connectivity of an n-node network

We first study the connectivity of a linear VANET with a fixed number of n nodes, and the results will be used to assist the analysis of network with dynamic number of nodes. To facilitate analysis, denote $Y_0 = X_{(1)}$. Prefixing $\mathbf{Y}$ with Y_0 leads to an extended distance vector, $\tilde{\mathbf{Y}} = [Y_0, \cdots, Y_{n-1}]^T$. The distribution of $\tilde{\mathbf{Y}}$ is presented in the following Lemma.

Lemma 3: The pdf of $\tilde{\mathbf{Y}}$ is $f_{\tilde{\mathbf{Y}}}(y_0, \cdots, y_{n-1}) = \frac{n!}{L^n}$, for positive y_m satisfying $\sum_{m=0}^{n-1} y_m \leq L$.

Proof: The extended distance vector can be expressed as a linear transformation of the vector, $\mathbf{X}_{(o)}$. It can be shown that the Jacobian of the transformation is 1. Thus, $f_{\tilde{\mathbf{Y}}}(\mathbf{y}) = f_{\mathbf{X}_{(o)}}(\mathbf{x})$, which is defined in Lemma 2. ■

Now we are in a position to present the connectivity probability of an n-node VANET.

Theorem 2: For a linear network with n nodes uniformly distributed over a section with length L, if the maximum transmission range of each node is d, then the probability that all the n nodes are connected is

$$P_n(d_0) = \sum_{m=0}^{k}(-1)^m\binom{n-1}{m}(1-md_0)^n, \quad d_0 \in \mathcal{L}_k(n-1), \quad k = 1, \cdots, n-1, \tag{19}$$

where $d_0 = d/L$ is the normalized transmission range.

Proof: The connectivity probability is proved with the assistance of the extended distance vector, $\tilde{\mathbf{Y}}$. It's obvious that when $Y_0 = X_{(1)} \geq L-d$, all the nodes are connected. Therefore, the connectivity probability given in (10) can be alternatively written as

$$P_n(d_0) = P_{n1} + P\left\{Y_0 \in [L-d, L]\right\}, \tag{20}$$

where P_{n1} is recursively defined as $P_{n1} \triangleq \int_0^{L-d} P_n\left(\frac{d}{L-y_0}\right)dy_0$. The conditional probability $P_n\left(\frac{d}{L-y_0}\right)$ can be expressed as

$$\begin{aligned} P_n\left(\frac{d}{L-y_0}\right) &= \underset{\mathbf{y}_{n-1}\in\mathcal{D}_{n-1}(d,L-y_0)}{\int\cdots\int} f_{\tilde{\mathbf{Y}}}(y_0,\cdots,y_{n-1})dy_1\cdots dy_{n-1}, \\ &= \frac{n!}{L^n}\int_0^{L-d}\text{Vol}\left[\mathcal{D}_{n-1}(d,L-y_0)\right]dy_0, \end{aligned} \tag{21}$$

where the second equality is based on Lemma 3.

To accommodate the partitioned definition range of $\text{Vol}\left[\mathcal{D}_{n-1}(d,L-y_0)\right]$, the integration limit, $[0, L-d]$, of (21), is partitioned into k consecutive sections as follows

$$[0, L-d] = [0, L-kd) \cup \left(\cup_{m=1}^{k-1}[L-(m+1)d, L-md)\right).$$

With the above partition and the results in Theorem 1, the probability P_{n1} when $d_0 \in \mathcal{L}_k(n-1)$ can be written as

$$P_{n1} = \frac{n!}{L^n}\left[\int_0^{L-kd}V_{n-1}^k(d,L-y_0)dy_0 + \sum_{m=1}^{k-1}\int_{L-(m+1)d}^{L-md}V_{n-1}^m(d,L-y_0)dy_0\right], \ d_0 \in \mathcal{L}_k(n-1).$$

The above integrals can be evaluated with the definition of $V_n^k(d,L)$, and the result when $d_0 \in \mathcal{L}_k(n-1)$ can be simplified to

$$P_{n1} = \frac{1}{L^n}\left[\sum_{m=0}^{k}(-1)^m\binom{n-1}{m}(L-md)^n - d^n\right]. \tag{22}$$

We next evaluate the probability, $P\left\{Y_0 \in [L-d, L]\right\} = \int_{L-d}^{L} f_{Y_0}(y_0)dy_0$. From order statistics, the pdf of $Y_0 = X_1$ is $f_{Y_0}(y_0) = \frac{n}{L^n}(L-y_0)^{n-1}$ [19], thus

$$P\left\{Y_0 \in [L-d, L]\right\} = d^n/L^n. \tag{23}$$

Combining (20), (22), and (23) leads to (19). ■

The result presented in Theorem 2 can be considered as the connectivity probability of a stationary network with n nodes uniformly distributed over a linear network.

B. Connectivity of Mobile Network

The connectivity probability of a mobile VANET is evaluated in this subsection.

Before proceeding to the connectivity probability, we have the following Lemma that will be used during the analysis.

Lemma 4:

$$\sum_{m=n+1}^{+\infty}\binom{m-1}{n}\frac{x^m}{m!} = -(-1)^n\frac{1}{n!}\gamma(n+1,-x), \tag{24}$$

where $\gamma(n,x) = \int_0^x t^{n-1}e^{-t}dt$ is the lower incomplete Gamma function.

Proof: Denote $F(x) = \sum_{m=n+1}^{+\infty}\binom{m-1}{n}\frac{x^m}{m!}$. Differentiating $F(x)$ with respect to x yields

$$F'(x) = \frac{1}{n!}\sum_{m=n+1}^{+\infty}\frac{x^{m-1}}{(m-1-n)!} = \frac{x^n}{n!}e^x. \tag{25}$$

The function $F(x)$ can be obtained by performing integration over $F'(x)$, and the result is

$$F(x) = F(0) - \frac{(-1)^n}{n!}\gamma(n+1, -x), \quad (26)$$

Since $F(0) = 0$, the proof is complete. ■

Theorem 3: Consider a linear network of length L. At any moment, the number of nodes in the network follows a Poisson distribution with parameter λ, and all the nodes uniformly distributed over $[0, L]$. Then the probability that all the nodes are connected all the time is

$$P_\lambda(d_0) = e^{-\lambda} - \sum_{n=0}^{k} \frac{e^{-\lambda}}{n!}\gamma(n+1, d_0 n\lambda - \lambda), \quad \text{for } d_0 \in \mathcal{L}_k(\infty), \quad k = 1, \cdots, \infty, \quad (27)$$

where $\mathcal{L}_k(\infty) = \left[\frac{1}{k+1}, \frac{1}{k}\right)$.

Proof: If $d_0 \in \mathcal{L}_k(\infty)$, then the connectivity probability, $P_\lambda(d_0) = \sum_{m=0}^{\infty} P_m(d_0) P\{N = m\}$, can be expressed as

$$P_\lambda(d_0) = e^{-\lambda} + e^{-\lambda}\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{n=0}^{\min(m-1,k)}(-1)^n\binom{m-1}{n}\alpha_n^m,$$

where $P_m(d_0)$ is given in Theorem 2, $\alpha_n = (1 - nd_0)\lambda$, and it's assumed that $P_0(d_0) = 1$. Exchanging the order of summation in the above expression, and noting the fact that $n \le k < m$, we have

$$P_\lambda(d_0) = e^{-\lambda} + e^{-\lambda}\sum_{n=0}^{k}(-1)^n\sum_{m=n+1}^{\infty}\binom{m-1}{n}\frac{\alpha_n^m}{m!}. \quad (28)$$

Combining (28) with Lemma 4 leads to (27). ■

The connectivity probability given in Theorem 3 is expressed as a function of key parameters of a mobile VANET, including node arrival rate λ_0, node mean speed μ_V, network length L, and node transmission range L.

V. Outage Transmission Range

In this section, we define a new measure, outage transmission range, to quantitatively identify the relationship between critical transmission range that can ensure network connectivity and key network parameters, such as node mobility and network density.

Definition 1: Define outage connectivity probability of a network as the probability that there is at least one node isolated from the remaining of the nodes in the network. ■

For stationary network with n nodes, the outage connectivity probability is $\epsilon_n = 1 - P_n(d_0)$; similarly, for mobile network, the outage connectivity probability is $\epsilon_\lambda = 1 - P_\lambda(d_0)$, with $P_n(d_0)$ and $P_\lambda(d_0)$ given in Theorems 2 and 3, respectively.

Definition 2: For a given outage connectivity probability ϵ, define outage transmission range, $\delta(\epsilon)$, as the value of the normalized transmission range, d_0, that satisfies $1 - P_n(d_0) = \epsilon$ for stationary network, or $1 - P_\lambda(d_0) = \epsilon$ for mobile network. ■

Since $P_n(d_0)$ and $P_\lambda(d_0)$ are continuous functions of d_0, $\delta(\epsilon)$ is well defined, *i.e.*, given ϵ and n (or λ), there exists k and $d_0 \in \mathcal{L}_k(n-1)$ (or $d_0 \in \mathcal{L}_k(\infty)$), such that $P_n(d_0) = 1-\epsilon$ (or $P_\lambda(d_0) = 1 - \epsilon$). In a practical system, it is extremely useful to identify the relationship between $\delta(\epsilon)$ and the network parameters n or λ. If n or λ are known, then each node in a VANET can set the transmission power corresponding to $\delta(\epsilon)$ with a small ϵ. Such a transmission range with small enough ϵ will ensure the connectivity of the entire network while minimizing interference among nodes. The solution of $\delta(\epsilon)$ is equivalent to finding the inverse function of $P_n(d_0)$ or $P_\lambda(d_0)$. Due to the summation terms involved in the expressions as shown in (19) and (27), it would be rather difficult, if not impossible, to find an exact closed-form expression for $\delta(\epsilon)$.

To simplify the solution, we resort to approximations of the connectivity probabilities by considering only the first two summation terms in the connectivity probability expressions given in (19) and (27), and the results are

$$\tilde{P}_n(d_0) = 1 - (n-1)(1-d_0)^n, , \quad (29a)$$

$$\tilde{P}_\lambda(d_0) = 1 - e^{-\lambda} + (d_0\lambda - \lambda + 1)e^{-d_0\lambda}, \quad (29b)$$

where the identities $\gamma(1,x) = 1 - e^{-x}$, $\gamma(2,x) = 1 - e^{-x} - xe^{-x}$, are used in derivation of (29b). It has been shown by simulation that the expressions in (29) provide very accurate approximations of the connectivity probabilities when $\epsilon \to 0$. For the special cases that $d_0 \in \mathcal{L}_1(\infty) \cup \mathcal{L}_2(\infty)$, the approximations are exact, *i.e.*, $\tilde{P}_n(d_0) = P_n(d_0)$, $\tilde{P}_\lambda(d_0) = P_\lambda(d_0)$, $\forall d_0 \in \mathcal{L}_1(\infty) \cup \mathcal{L}_2(\infty)$.

Solving (29) in d_0 leads to

$$\delta_n(\epsilon) \approx 1 - \left(\frac{\epsilon}{n-1}\right)^{1/n}. \quad (30a)$$

$$\delta_\lambda(\epsilon) \approx 1 - \frac{1}{\lambda} - \frac{1}{\lambda}W\left(\left(\epsilon - e^{-\lambda}\right)e^{\lambda-1}\right), \quad (30b)$$

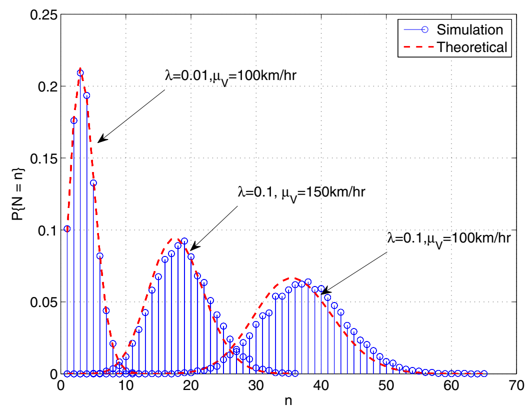
where $W(x)$ is the Lambert's W function defined as the inverse of $f(x) = xe^x$.

The results in (30) present approximated expressions of the outage transmission ranges as a function of the number of nodes n in a stationary VANET, or as a function of network parameter $\lambda = L\lambda_0/\mu_V$ in a mobile VANET. The results can be used by nodes in a VANET to determine their respective transmission powers to ensure network connectivity as well as to minimize mutual interference. Simulation results show that the expressions in (30) provide very accurate approximations of the actual outage transmission range.

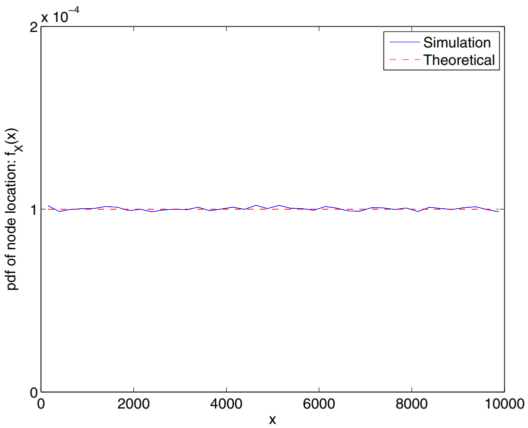
VI. Numerical Examples

Numerical examples are provided in this section to verify the theoretical results presented in this paper, as well as to investigate the impacts of various network parameters on the connectivity of VANETs.

We first investigate the accuracy of the mobility model presented in Section II. Fig. 1 shows the steady state distribution of node population and node location under various values of node arrival rate, λ_0, and node mean speed, μ_V. In this example, the random node speeds are generated by following a WSS Gaussian random process with exponential auto-correlation function. The theoretical curves are generated



(a) Distribution of number of nodes



(b) Distribution of node location

Fig. 1. Verification of mobility model as $t \to \infty$.

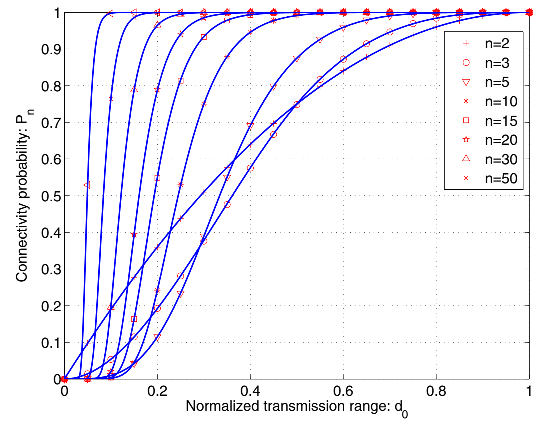


Fig. 2. Connectivity probability for stationary networks with fixed number of n nodes (markers: simulation results, lined curves: theoretical results).

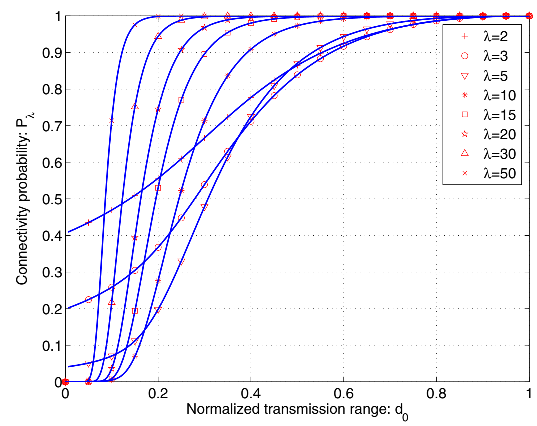


Fig. 3. Connectivity probability for mobile networks (markers: simulation results, lined curves: theoretical results).

by using the mobility model results presented in Lemma 1. Comparison between the simulation results and theoretical results reveals that the mobility model renders a very accurate representation of the steady state distribution of random node population and location.

Fig. 2 shows the connectivity probability of stationary networks with fixed number of nodes as a function of normalized transmission range, $d_0 = \frac{d}{L}$. The marks are obtained through empirical simulation, and the curves are calculated through theoretical results presented in Theorem 2. Perfect match is observed between the simulation results and the theoretical results under all system configurations. It is observed that when n is large, the transition from 0 connectivity to 100% connectivity requires only a small variation in d_0. Thus, the outage transmission range, $\delta(\epsilon)$, tends to a constant for $0 < \epsilon < 1$, and we call this transmission range as the critical transmission range, *i.e.* with $n \to \infty$, $P_n \to 0$ if $d_0 < \delta(\epsilon)$, while $P_n \to 1$ if $d_0 > \delta(\epsilon)$.

The connectivity probability of a mobile VANET with random number of nodes is shown in Fig. 3 under various values of Poisson parameter, λ. Again, perfect agreement is observed between the simulation results (marks) and analytical results (curves), and this validates the theoretical result presented in Theorem 3. When P_λ is large, network with larger λ requires less transmission range to achieve a fixed value of P_λ. Since λ is proportional to node arrival rate, and inverse proportional to node speed, increasing node arrival rate or reducing node speed will increase network connectivity under a fixed normalized transmission range. It should be noted that when λ is small, $P_\lambda(0) > 0$. This non-zero probability is contributed by the cases when there are less than two nodes in the network, and the network is defined as connected in such situation.

Fig. 4 compares the performance of stationary networks and mobile networks with the same average number of nodes in network. It can be seen that when the connectivity probability is large, the stationary network always outperforms mobile network with the same average numbers of nodes in network. The difference between the two networks gradually diminishes with the increase of n or λ.

The outage transmission ranges, $\delta(\epsilon)$, for stationary network and mobile network are shown as a function of n and λ in Figs. 5(a) and 5(b), respectively. The curves are obtained through the approximated expressions in (29), while the marks are the exact values obtained through numerical solution of the results

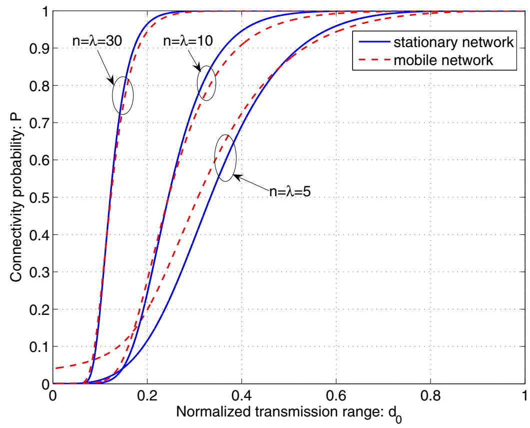
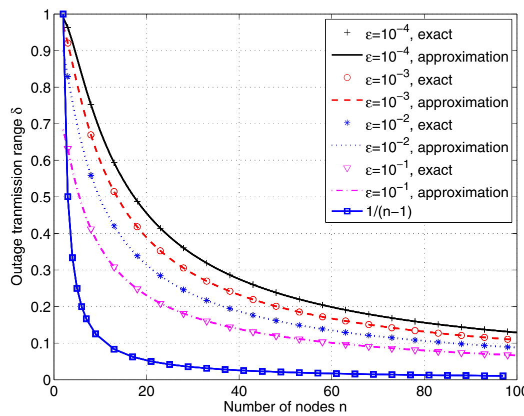
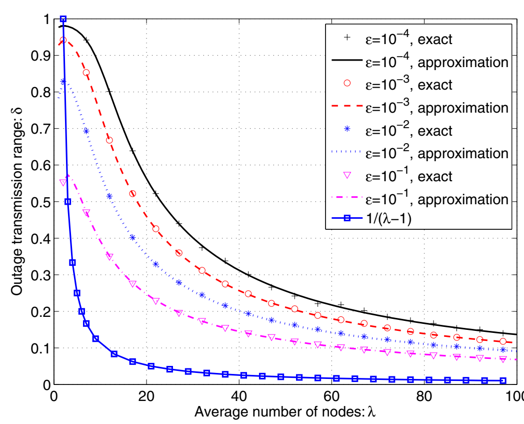


Fig. 4. Comparison of connectivity probability of stationary network and mobile network.



(a) Stationary Network



(b) Mobile Network

Fig. 5. Outage transmission distance

in Theorems 2 and 3. Perfect matches are observed between the exact results and the approximations when ϵ is small. Therefore, the expressions in (29) provide a very accurate approximation of the outage transmission range, and the simple expressions can be used to determine the transmission range of each node in the network. As expected, a relatively longer transmission range is required for smaller outage probability. As a reference, the curve, $\delta = 1/(n-1) = 1/(\lambda-1)$, which corresponds to the transmission range of a deterministic network with n nodes evenly distributed from $[0, L]$ is also shown in the figure. It can be seen that when n is large, network with random node locations always require a longer transmission range compared to the deterministic network with optimum node distribution.

VII. Conclusions

The network connectivity of a linear VANET with high speed mobile nodes and dynamic node populations was investigated. A new mobility model was developed to represent the steady state distributions of both node population and node location. The statistical properties of network connectivity were investigated with the help of the new mobility model and the volume of a hyperplane intersected n-cube. The impacts of key network parameters, such as node arrival rate, random node speed, node population distribution, and network length, are incorporated into exact closed-form expressions of connectivity probabilities of mobile VANET. Numerical examples demonstrated that the new mobility model along with the analytical connectivity results accurately capture the behaviors of a mobile VANET with rapidly changing network topologies and node densities. Results from numerical examples indicated that node mobility will generally degrade network connectivity if we do not utilize the storage capabilities of the intermediate node. The performance of mobile network and stationary network gradually converges with the increase of node density. In addition, when node density is large, a small variation in transmission rage leads to significant differences in connectivity probability. The results in this paper are also applicable to bi-directional highway.

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