Approximating a Sum of Random Variables with a Lognormal

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***Abstract*— A simple, novel, and general method is presented in this paper for approximating the sum of independent or arbitrarily correlated lognormal random variables (RV) by a single lognormal RV. The method is also shown to be applicable for approximating the sum of lognormal-Rice and Suzuki RVs by a single lognormal RV. A sum consisting of a mixture of the above distributions can also be easily handled. The method uses the moment generating function (MGF) as a tool in the approximation and does so without the extremely precise numerical computations at a large number of points that were required by the previously proposed methods in the literature. Unlike popular approximation methods such as the Fenton-Wilkinson method and the Schwartz-Yeh method, which have their own respective short-comings, the proposed method provides the parametric flexibility to accurately approximate different portions of the lognormal sum distribution. The accuracy of the method is measured both visually, as has been done in the literature, as well as quantitatively, using curve-fitting metrics. An upper bound on the sensitivity of the method is also provided.**

***Index Terms*— lognormal distribution, correlation, Suzuki distribution, lognormal-Rice distribution, moment methods, characteristic function, moment generating function, approximation methods, co-channel interference.**

I. Introduction

THE attenuation due to shadowing in wireless channels is often modeled by the lognormal distribution [1], [2]. Hence, in the analysis of wireless systems, one often encounters the sum of lognormal random variables (RV). For example, it characterizes the total co-channel interference (CCI) power from all the transmissions in neighboring cells. The lognormal distribution is also of interest in outage probability analysis [2, Chp. 3] and in ultra wide band systems [3]. Given the importance of the lognormal sum distribution in wireless communications as well as in other fields such as optics and reliability theory, considerable efforts have been devoted to analyze its statistical properties. While exact closed-form expressions for the lognormal sum probability distribution functions (PDF) are unknown, several analytical approximation methods have been proposed in the literature [4]–[9].

The methods proposed in the literature can be broadly classified into two categories. The methods by Fenton-Wilkinson (F-W) [4], Schwartz-Yeh (S-Y) [5], and Beaulieu-Xie [6] approximate the lognormal sum by a single lognormal RV. The permanence of the lognormal PDF lends further credence to these methods [10], [11]. The methods by Farley [2], [5], Ben Slimane [7], and Schleher [8] instead compute a compound distribution based on the properties of the lognormal RV. The compound distribution can be specified in several ways. For example, the methods in [5], [7] specify the approximating distribution in terms of strict lower bounds of the cumulative distribution function (CDF), while [8] partitions the range of the lognormal sum into three segments, with each segment being approximated by a distinct lognormal RV.

Beaulieu *et al.* [6], [12] have studied in detail the accuracy of several of the above methods, and have shown that all the methods have their own advantages and disadvantages – none is unquestionably better than the others. The F-W method is inaccurate for estimating the CDF for small values of the argument, while the S-Y method is inaccurate for estimating the complementary CDF (CCDF) for large values of the argument. The Farley's method and, more generally, the formulae derived in [7] are strict bounds on the CDF that can be loose approximations for certain typical parameters of interest. The methods also differ considerably in their complexity. For example, the S-Y method involves solving non-linear equations and requires an iterative procedure to handle the sum of more than two RVs. Only the F-W method offers a closed-form solution for calculating the underlying parameters of the approximating lognormal PDF.

That the MGF (CF) of a sum of independent RVs can be written as the product of the MGFs (CFs) of the individual RVs [13] is another property that has been exploited by methods proposed in the literature [6], [11]. However, as we discuss below, the methods require extremely accurate numerical computation at a sufficiently large number of points and are quite involved.[1] Moreover, the CF-based methods proposed so far are fundamentally limited to the case in which the lognormal RVs are independent.

Barakat [11] applied an inverse Fourier transform to the product of the lognormal CFs to determine the PDF of lognormal sum. The individual lognormal CFs were computed numerically using a Taylor series expansion. However, the oscillatory property of the Fourier integrand as well as the slow decay rate of the lognormal PDF tail, made the numerical eval-

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[1]While the CF is a special case of the MGF, we choose to treat the two as separate to keep the discussion clear.

uation difficult and inaccurate [6]. Also, given the numerical approach, no analytical expressions of the approximate distribution were provided. A similar approach was also suggested by Anderson [14]. Beaulieu-Xie's [6] elegant and conceptually simple method calculates the composite CDF by numerically evaluating the inverse Fourier transform of the lognormal sum at several points. The very high numerical precision required is achieved using a modified Clenshaw-Curtis method. The composite CDF is then plotted on 'lognormal paper', in which the lognormal PDF appears as a straight line. The parameters of the approximating lognormal distribution are determined by minimizing the maximum (minimax) error in a given interval. While the method is optimal in the minimax sense on lognormal paper, this does not imply optimality in directly matching the probability distribution.

This paper makes the following contributions. First, we present a general method that uses the MGF as a tool to approximate the distribution of a sum of *independent lognormal* RVs by a single lognormal RV. The method is motivated by an interpretation of the metrics used by the F-W and S-Y methods as weighted integrals of the PDF. By using an approximate and short Gauss-Hermite expansion of the lognormal MGF, the proposed method circumvents the requirement for very precise numerical computations at a large number of points. It is not recursive, it is numerically stable and, as we show, very accurate. The method also offers considerable flexibility compared to previous approaches in matching different regions of the probability distribution.

Second, we show that our method is also a powerful tool for accurately approximating the sum of *correlated lognormal* RVs by a single lognormal RV. Third, we show that the proposed method is comprehensive enough to also approximate – by a lognormal RV – the sum of independent *Suzuki* RVs [15] and, more generally, the sum of *lognormal-Rice* RVs. It does so more accurately than previously proposed methods, as we discuss later. The proposed method can also handle the sum of a *mixture* of lognormal RVs, Suzuki RVs, and lognormal-Rice RVs. Finally, we compare the accuracy of the proposed lognormal approximation method with others using curve-fitting metrics defined over a region of interest. A general sensitivity analysis is also provided in the paper to study the impact of errors or changes in parameters on the accuracy of the method, and an upper bound for the sensitivity is derived. The proposed method has applications in spectral efficiency analysis of cellular systems [16], co-channel interference modeling, determining cell coverage in interference-limited cells [2], and signal outage probability evaluation in dispersive environments when different multipaths undergo different shadowing. Another useful application is cooperative networks in which the channels between the relays and the source/destination have different shadowing gains [17].

The paper is organized as follows: Section II reviews the lognormal sum approximation methods in the literature and makes a key observation about their behaviors. Section III motivates and defines the method proposed in this paper for the case of independent lognormal RVs. Section IV handles the case of the sum of correlated lognormal RVs, and Section V handles the sum of Suzuki or lognormal-Rice RVs. Numerical examples are used in Section VI to compare it with other methods and to demonstrate its accuracy. Section VII considers the accuracy of lognormal sum approximation methods in a specified region of interest, and derives an upper bound for the sensitivity of the method. The conclusions follow in Section VIII.

II. Understanding Lognormal Sum Approximation Methods

Let $Y_1, \ldots, Y_K$ be K independent, but not necessarily identical, lognormal RVs with PDFs denoted by $p_{Y_k}(x)$, for $1 \leq k \leq K$, respectively. Then each Y_k can be written as $10^{0.1 X_k}$ such that X_k is a Gaussian random variable with mean μ_{X_k} dB and standard deviation σ_{X_k} dB, *i.e.*, $X_k \sim \mathcal{N}(\mu_{X_k}, \sigma^2_{X_k})$. Since the K lognormal RVs are independently distributed, the PDF of the lognormal sum $\sum_{k=1}^K Y_k$ is given by

$$p_{\left(\sum_{k=1}^K Y_k\right)}(x) = p_{Y_1}(x) \otimes p_{Y_2}(x) \otimes \ldots \otimes p_{Y_K}(x), \tag{1}$$

where $\otimes$ denotes the convolution operation.

General closed-form expressions for the sum PDF are not known. However, it has been recognized that the lognormal sum can be well approximated by a new lognormal RV $Y = 10^{0.1X}$, where X is a Gaussian RV with mean μ_X and variance σ^2_X. The pdf of Y takes the form

$$p_Y(y) = \frac{\xi}{y\sigma_X\sqrt{2\pi}} \exp -\frac{(\xi \log_e y - \mu_X)^2}{2\sigma^2_X}, \tag{2}$$

where $\xi = 10/\log_e 10$ is a scaling constant. Thus, the problem is now equivalent to estimating the lognormal moments μ_X and σ^2_X given the corresponding statistics of the constituent lognormal RVs, $\{Y_k\}_{k=1}^K$.

The Fenton-Wilkinson (F-W) method computes the values of μ_X and σ^2_X by exactly matching the first and second central moments of Y with those of $\sum_{k=1}^K Y_k$:

$$\int_0^\infty y p_Y(y) dy = \sum_{k=1}^K \int_0^\infty y p_{Y_k}(y) dy, \tag{3a}$$

$$\int_0^\infty (y - \mu_Y)^2 p_Y(y) dy = \sum_{k=1}^K \int_0^\infty (y - \mu_{Y_k})^2 p_{Y_k}(y) dy, \tag{3b}$$

where μ_Y and μ_{Y_k} are the means of Y and Y_k, respectively. While the F-W method accurately models the *tail portion* (large values of Y) of the lognormal sum PDF, it is quite inaccurate near the *head portion* (small values of Y) of the sum PDF, especially for large values of σ_{X_k} [12]. Since the F-W method computes the logarithmic moments μ_X and σ_X by matching the linear moments μ_Y and σ_Y, the mean square error in μ_X and σ_X increases with a decrease in the spread of the mean values or an increase in the spread of the standard deviations of the summands [18]. The method breaks down for $\sigma_{X_k} > 4$ dB when it tries to model the behavior of $10\log_{10}\left(\sum_{k=1}^K Y_k\right)$ [2].

The Schwartz-Yeh (S-Y) method instead matches the moments in the log-domain, *i.e.*, it equates the first and second central moments of $10\log_{10} Y$ with those of $10\log_{10}(\sum_{k=1}^{K} Y_k)$:

$$\int_0^\infty (\log_{10} y)\, p_Y(y)dy = \int_0^\infty (\log_{10} y)\, p_{\left(\sum_{k=1}^{K} Y_k\right)}(y)dy, \tag{4a}$$

$$\int_0^\infty (10\log_{10} y - \mu_X)^2 p_Y(y)dy = \int_0^\infty (10\log_{10} y - \mu_{X'})^2 p_{\left(\sum_{k=1}^{K} Y_k\right)}(y)dy, \tag{4b}$$

where μ_X and $\mu_{X'}$ are the mean values of $X = 10\log_{10} Y$ and $X' = 10\log_{10}\sum_{k=1}^{K} Y_k$, respectively. While the match is exact for $K = 2$, an approximate iterative technique needs to be used for $K > 2$. The unknowns μ_X and σ_X are evaluated numerically. The S-Y method is more involved than the F-W method because the expectation of the logarithm sum cannot be directly written in terms of the moments of the summands. As mentioned, the S-Y method is inaccurate near the tail portion of the distribution function and can significantly underestimate small values of the CCDF [12].

Since the moments can be interpreted as weighted integrals of the PDF, both the F-W method and the S-Y method can be generalized by the following system of equations:

$$\int_0^\infty w_i(y)p_Y(y)dy = \int_0^\infty w_i(y)p_{\left(\sum_{k=1}^{K} Y_k\right)}(y)dy, \quad \text{for } i = 1 \text{ and } 2. \tag{5}$$

The F-W method uses the weight functions $w_1(y) = y$ and $w_2(y) = (y - \mu_Y)^2$, both of which monotonically increase with y. Thus, approximation errors in the tail portion of the sum PDF are penalized more. This explains why the F-W method tracks the tail portion well. On the other hand, the S-Y method employs the weight function $w_1(y) = \log_{10} y$ and $w_2(y) = (10\log_{10} y - \mu_X)^2$. Due to the singularity of $\log_{10} y$ at $y = 0$, mismatches near the origin are severely penalized by both these weight functions. Compared to the F-W method, the S-Y method also accords a lower penalty to errors in the PDF tail. For these reasons, it does a better job tracking the head portion of the distribution function. However, both the F-W and the S-Y methods use fixed weight functions and offer no way of overcoming their respective shortcomings.

Similarly, Schleher's cumulants matching method [8] accords polynomially increasing penalties to the approximation error in the tail portion of the PDF. This is because the first three cumulants are, in effect, the first three central moments. By plotting the x-axis in dB scale on lognormal paper, the Beaulieu-Xie method also gives more weight to the tail portion.

The weighted integral interpretations of these approximation methods motivates the flexible and simple lognormal sum approximation method proposed in the next section that also exploits the desirable properties of the MGF.

III. Lognormal Sum Approximation Using Gauss-Hermite Expansion of MGF

A. Motivation

The simplicity of the F-W method arises from the fact that the mean and variance of a sum of independent RVs can be written directly as the sum of the mean and variance of the individual RVs. The MGF of the sum of independent RVs also possesses this desirable property, in that it can be written directly in terms of the MGFs of the individual RVs.

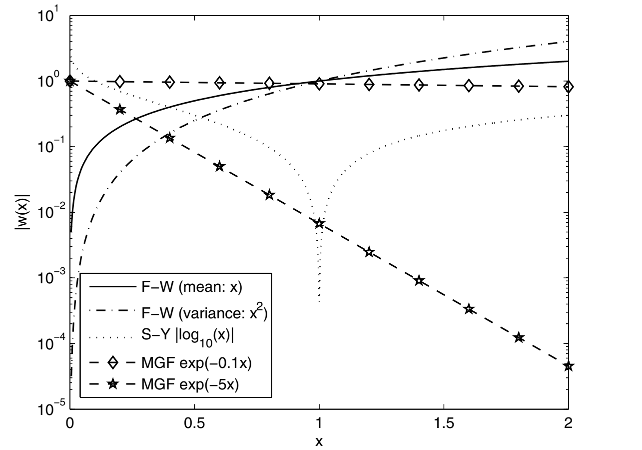


Fig. 1. Weight functions employed by F-W, S-Y, and the proposed MGF-based method.

The MGF of the RV Y is defined as

$$\Psi_Y(s) = \int_0^\infty \exp(-sy)p_Y(y)dy. \tag{6}$$

It can be seen from (6) that the MGF can also be interpreted as the weighted integral of the PDF $p_Y(y)$, with the weight function being the exponential function $\exp(-sy)$, which monotonically decreases (in y) for real and positive values of s. Varying $\mathrm{Re}(s)$ from 0 to ∞ provides a mechanism for adjusting, as required, the penalties allocated to errors in the head and tail portions of the sum PDF. Figure 1 compares the absolute values of the various weight functions discussed above, in log-scale. Moreover, since the lognormal RVs $\{Y_k\}_{k=1}^{K}$ are independently distributed, the MGF of the lognormal sum $\sum_{k=1}^{K} Y_k$ can be written as

$$\Psi_{\left(\sum_{k=1}^{K} Y_k\right)}(s) = \prod_{k=1}^{K} \Psi_{Y_k}(s). \tag{7}$$

Based on the discussion above, we can see that the MGF possesses two desirable properties. First, the MGF is a weighted integral of the PDF with an adjustable parameter, s. Second, the MGF of the sum PDF can be easily expressed as the product of the MGFs of the individual independent RVs. These two properties render the MGF as a preferable candidate for the lognormal sum approximation problem, as we show below.

B. MGF-based Lognormal Sum Approximation

The development of the MGF-based lognormal sum approximation method requires a closed-form expression for the MGF of lognormal RV. While no general closed-form expression for the lognormal MGF is available, for real s, it can be readily expressed by a series expansion based on

Gauss-Hermite integration.[2] The MGF of a lognormal RV Y for real s can be written as

$$\begin{aligned}\Psi_Y(s) &= \int_0^{\infty} \exp(-sy) \frac{\xi}{y\sigma_X\sqrt{2\pi}} \exp\left[-\frac{(\xi \log_e y - \mu_X)^2}{2\sigma_X^2}\right] dy, \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \exp\left[-s\exp\left(\frac{\sqrt{2}\sigma z + \mu}{\xi}\right)\right] \exp(-z^2) dz, \\ &= \sum_{n=1}^{N} \frac{w_n}{\sqrt{\pi}} \exp\left[-s\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi}\right)\right] + R_N,\end{aligned} \quad (8)$$

where μ_X and σ_X are the mean and standard deviation of the Gaussian RV $X = 10\log_{10} Y$. The final expression is the Gauss-Hermite series expansion of the MGF function, N is the Hermite integration order, and R_N is a remainder term that decreases as N increases. The weights, w_n, and abscissas, a_n, for N up to 20 are tabulated in [20, Tbl. 25.10]. From it, we can define the Gauss-Hermite representation of the MGF by removing R_N as follows:

$$\widehat{\Psi}_Y(s;\mu_X,\sigma_X) \triangleq \sum_{n=1}^{N} \frac{w_n}{\sqrt{\pi}} \exp\left[-s\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi}\right)\right]. \quad (9)$$

The lognormal sum $\sum_{k=1}^{K} Y_k$ can now be approximated by a lognormal RV $Y = 10^{0.1X}$, where $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, by matching the MGF of Y with the MGF of the lognormal sum $\sum_{k=1}^{K} Y_k$ at two different, real and positive values of s, namely, s_1 and s_2. This sets up the following system of two independent equations to calculate μ_X and σ_X^2:

$$\sum_{n=1}^{N} \frac{w_n}{\sqrt{\pi}} \exp\left[-s_i\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi}\right)\right] = \prod_{k=1}^{K} \widehat{\Psi}_{Y_k}(s_i;\mu_{X_k},\sigma_{X_k}), \quad \text{for } i = 1 \text{ and } 2, \quad (10)$$

where μ_{X_k} and σ_{X_k} are the known lognormal moments of the lognormal RV $Y_k = 10^{0.1X_k}$, *i.e.*, $X_k \sim \mathcal{N}(\mu_{X_k}, \sigma_{X_k}^2)$. Note that the right hand side of the above two equations is a constant number that needs to be calculated only once. These non-linear equations in μ_X and σ_X can be readily solved numerically using standard functions such as `fsolve` in Matlab and `NSolve` in Mathematica.

Better estimates of μ_X and σ_X are obtained by increasing the Hermite integration order N; on the other hand, reducing N decreases the computational complexity. We have found $N = 12$ to be sufficient to accurately determine the values of μ_X and σ_X; this is small compared to the 20-40 terms required to achieve numerical accuracy in the S-Y method [21]. Furthermore, unlike the S-Y method, no iteration in K is required – the right hand side of (10) can be computed right at the beginning of the method at $s = s_1$ and $s = s_2$.

Most importantly, as highlighted before, the penalty for PDF mismatch can be adjusted by choosing s appropriately. Increasing s penalizes more the errors in approximating the head portion of the sum PDF, while reducing s penalizes errors in the tail portion, as well. The inevitable trade-off that needs to be made in approximating both the head and tail portions of the PDF, can now be done depending on the application. For example, when the lognormal sum arises because various signal components add up [3], the main performance metric is the outage probability. For this, the head of the CDF needs to be computed accurately. On the other hand, when the lognormal sum appears as a denominator term, for example, when the powers from co-channel interferers add up in the signal to noise plus interference ratio calculation, it is the tail portion of the CCDF that needs to be calculated accurately. The proposed method can handle both of these applications by using different matching pairs (s_1, s_2). Guidelines for choosing (s_1, s_2) are elaborated upon in Sections VI and VII.

IV. Sum of Correlated Lognormal Random Variables

Correlated lognormal RVs often arise in cellular systems because the shadowing of inter-cell interferers is correlated with a typical site-to-site correlation coefficient of 0.5 [22], [23]. The correlated sum case has been investigated in [9], [24]–[26], and extensions to the F-W [24], [25], S-Y [26], and Cumulants [24] methods have been proposed to handle it. But, Farley's method, the Beaulieu-Xie method, and the bounds in [7] do not apply to the sum of correlated lognormal RVs. Outage probability bounds, which, in effect, specify a compound distribution, are derived in [9] using the arithmetic-geometric mean inequality and can handle the correlated sum case. However, the basic limitations of the various methods still apply – the S-Y extension cannot accurately estimate small values of the CCDF [27][3], the F-W extension again cannot accurately estimate small values of the CDF, and the bounds are loose for larger logarithmic variances.

We now consider the general case of K correlated lognormal RVs, $\{Y_k\}_{k=1}^{K}$, with corresponding Gaussian RVs, $\{X_k\}_{k=1}^{K}$, which have an arbitrary correlation matrix $\mathbf{C}$. We derive the set of two equations that will yield the parameters for the approximating lognormal RV.

When K lognormal RVs, $\{Y_k\}_{k=1}^{K}$, are correlated, the corresponding Gaussian RVs, $X_k = 10\log_{10} Y_k$ follow the joint distribution

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{K/2}|\mathbf{C}|^{1/2}} \exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu})^\dagger \mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2}\right), \quad (11)$$

where $\mathbf{C}$ is the covariance matrix and $\boldsymbol{\mu}$ is the vector of means of the Gaussian RVs. The MGF of $Y_1 + \cdots + Y_K$ can then be written as:

$$\Psi^{(c)}_{\left(\sum_{k=1}^{K} Y_k\right)}(s) = \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{K/2}|\mathbf{C}|^{1/2}} \prod_{k=1}^{K} \exp\left(-s\left[\exp\left(\frac{x_k}{\xi}\right)\right]\right) \times \exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu})^\dagger \mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2}\right) d\mathbf{x}, \quad (12)$$

[2] Naus [19] has derived a formula for the MGF of the sum of two lognormal RVs. While the formula can be extended to handle the sum of an even number of lognormal RVs, it only applies to the special case of an even number of identical and independent RVs, and is in the form of an infinite series.

[3] The outage probability was used in [27] to compare the different methods.

where $|.|$ denotes the determinant and $(.)^\dagger$ denotes the Hermitian transpose.

Let $\mathbf{C}_{\text{sq}}$ be the square root of the correlation matrix $\mathbf{C}$, *i.e.*, $\mathbf{C} = \mathbf{C}_{\text{sq}}\mathbf{C}_{\text{sq}}^\dagger$. In general, let the eigen-decomposition of $\mathbf{C}$ be $\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\dagger$, where $\mathbf{U}$ is the eigen-space of $\mathbf{C}$ and the diagonal matrix $\boldsymbol{\Lambda}$ contains the eigenvalues of $\mathbf{C}$. Then $\mathbf{C}_{\text{sq}} = \mathbf{U}\boldsymbol{\Lambda}^{1/2}$. When the decorrelating transformation $\mathbf{x} = \sqrt{2}\mathbf{C}_{\text{sq}}\mathbf{z} + \boldsymbol{\mu}$ is used, x_k is given by

$$x_k = \sqrt{2}\sum_{j=1}^{K} c'_{kj} z_j + \mu_k, \quad k = 1, \dots, K, \tag{13}$$

where c'_{kj} is the $(k, j)^{\text{th}}$ element of $\mathbf{C}_{\text{sq}}$. Therefore, the MGF equation becomes

$$\Psi^{(c)}_{\left(\sum_{k=1}^K Y_k\right)}(s) = \int_{-\infty}^{\infty}\!\!\cdots\!\int_{-\infty}^{\infty} \frac{1}{\pi^{K/2}} \prod_{k=1}^{K} \exp\left(-s\left[\exp\left(\frac{\sqrt{2}}{\xi}\sum_{j=1}^{K} c'_{kj} z_j + \frac{\mu_k}{\xi}\right)\right]\right) \times \exp\left(-\mathbf{z}^\dagger \mathbf{z}\right) dz_1 dz_2 \dots dz_K. \tag{14}$$

Taking the Gauss-Hermite expansion with respect to z_1 yields

$$\Psi^{(c)}_{\left(\sum_{k=1}^K Y_k\right)}(s) = \int_{-\infty}^{\infty}\!\!\cdots\!\int_{-\infty}^{\infty} \frac{1}{\pi^{(K-1)/2}} \exp\left(-\sum_{i=2}^{K} |z_i|^2\right) \sum_{n_1=1}^{N} \frac{w_{n_1}}{\sqrt{\pi}} \times \prod_{k=1}^{K} \exp\left(-s\left[\exp\left(\frac{\sqrt{2}}{\xi}\sum_{j=2}^{K} c'_{kj} z_j + \frac{\sqrt{2}}{\xi} c'_{k1} a_{n_1} + \frac{\mu_k}{\xi}\right)\right]\right) \times dz_2 \dots dz_K + R_N^{(1)}, \tag{15}$$

where $R_N^{(1)}$ is a remainder term that decreases as N increases. Proceeding in a similar manner for $z_2, \dots, z_K$, we get

$$\Psi^{(c)}_{\left(\sum_{k=1}^K Y_k\right)}(s) = \sum_{n_K=1}^{N} \cdots \sum_{n_1=1}^{N} \frac{w_{n_1} \dots w_{n_K}}{\pi^{K/2}} \times \prod_{k=1}^{K} \exp\left(-s\left[\exp\left(\frac{\sqrt{2}}{\xi}\sum_{l=1}^{K} c'_{kl} a_{n_l} + \frac{\mu_k}{\xi}\right)\right]\right) + R_N^{(K)}, \tag{16}$$

where $R_N^{(K)}$ is the remainder term. Rearranging the terms and dropping the remainder term result in the following definition of the MGF approximation function $\widehat{\Psi}^{(c)}_{\left(\sum_{k=1}^K Y_k\right)}(s; \boldsymbol{\mu}, \mathbf{C})$:

$$\widehat{\Psi}^{(c)}_{\left(\sum_{k=1}^K Y_k\right)}(s; \boldsymbol{\mu}, \mathbf{C}) \triangleq \sum_{n_1=1}^{N} \cdots \sum_{n_K=1}^{N} \left[\prod_{k=1}^{K} \frac{w_{n_k}}{\sqrt{\pi}}\right] \times \exp\left(-s\sum_{k=1}^{K}\left[\exp\left(\frac{\sqrt{2}}{\xi}\sum_{j=1}^{K} c'_{kj} a_{n_j} + \frac{\mu_k}{\xi}\right)\right]\right). \tag{17}$$

Therefore, the sum, $Y_1 + \cdots + Y_K$, of K correlated lognormal RVs can be approximated by a single lognormal RV, $Y = 10^{0.1X}$, using the following two equations:

$$\widehat{\Psi}_Y(s_i; \mu_X, \sigma_X) = \widehat{\Psi}^{(c)}_{\left(\sum_{k=1}^K Y_k\right)}(s_i; \boldsymbol{\mu}, \mathbf{C}), \quad \text{at } i = 1 \text{ and } 2, \tag{18}$$

where $\widehat{\Psi}^{(c)}_{\left(\sum_{k=1}^K Y_k\right)}(s; \boldsymbol{\mu}, \mathbf{C})$ is given by (17) and $\widehat{\Psi}_Y(s; \mu_X, \sigma_X)$ is given by (9). The value of $N = 12$ was found to be accurate for the correlated case, as well.

For the special case of the sum of two zero-mean lognormal RVs with correlation coefficient ρ and variance σ dB, the MGF approximation function can be written in closed-form in terms of ρ as

$$\widehat{\Psi}^{(c)}_{(Y_1+Y_2)}(s; \dots) \triangleq \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{w_n w_m}{\pi} \exp\left(-s\left[\exp\left(\frac{\sqrt{2}\sigma a_m}{\xi}\right) + \exp\left(\frac{\sqrt{2(1-\rho^2)}\sigma a_n + \sqrt{2}\sigma\rho a_m}{\xi}\right)\right]\right). \tag{19}$$

V. Sum of Independent Suzuki or Lognormal-Rice Random Variables

The Suzuki RV is a product of a lognormal RV and a Rayleigh fading RV. When a line-of-sight component is also present, we instead get a lognormal-Rice RV, which is a product of a lognormal RV and a Ricean-fading RV, and can be written as

$$W = Z\ 10^{0.1X}, \tag{20}$$

where Z is a Ricean RV with unit power and Rice-coefficient κ. The lognormal-Rice PDF takes the integral form

$$p_W(w) = \int_0^{\infty} \frac{2w(\kappa+1)}{y^2} \exp\left(-\kappa - (\kappa+1)\frac{w^2}{y^2}\right) \times I_0\left(\frac{2w}{y}\sqrt{\kappa(\kappa+1)}\right) p_Y(y) dy, \tag{21}$$

where $Y = 10^{0.1X}$ has the lognormal probability distribution given by (2). Setting $\kappa = 0$ results in a Suzuki distribution.

Sums of lognormal-Rice or Suzuki RVs arise, for example, when the short-term fading is also taken into account in the co-channel interference power calculation or in the calculation of the total instantaneous power received in a frequency-selective channel, when the multipaths undergo independent Ricean/Rayleigh fading and lognormal shadowing.

To approximate the sum of these RVs by a lognormal, an extension of the F-W-based moment matching technique was proposed in [28]. Another technique is a two-step approximation process in which each of the lognormal-Rice or Suzuki RVs is first approximated by a lognormal RV (by equating the means and variances), and then the sum of the lognormal RVs is again approximated by a single lognormal RV using the F-W or the S-Y methods. The sum of Suzuki RVs has also been approximated by another Suzuki RV in [29]. Exact formulae are available in the literature that express the outage probability of a sum of lognormal-Rice RVs in the form of a single integral, which is evaluated numerically [30], [31]. However, these do not address the problem of approximating the sum by a single lognormal RV.

The method proposed in the previous sections applies to the sum of lognormal-Rice RVs as follows. Using Gauss-Hermite integration and neglecting the remainder term results in the following MGF approximation for the k^{th} RV [32]

$$\widehat{\Psi}_{S_k}(s;\mu_k,\sigma_k,\kappa_k) \triangleq \sum_{n=1}^{N} \frac{w_n(1+\kappa_k)/\sqrt{\pi}}{1+\kappa_k+s\exp\left(\frac{\sqrt{2}\sigma_k a_n}{\xi}+\frac{\mu_k}{\xi}\right)} \times \exp\left(-\frac{s\kappa_k \exp\left(\frac{\sqrt{2}\sigma_k a_n}{\xi}+\frac{\mu_k}{\xi}\right)}{1+\kappa_k+s\exp\left(\frac{\sqrt{2}\sigma_k a_n}{\xi}+\frac{\mu_k}{\xi}\right)}\right), \quad (22)$$

where μ_k and σ_k are the logarithmic mean and logarithmic standard deviation of the shadowing component, and κ_k is the Rice factor of the k^{th} summand.

Therefore, the sum of K lognormal-Rice RVs, $S_1+\cdots+S_K$, can be approximated by a single lognormal RV, $Y=10^{0.1X}$, by the following two equations

$$\widehat{\Psi}_Y(s_i;\mu_X,\sigma_X)=\prod_{k=1}^{K}\widehat{\Psi}_{S_k}(s_i;\mu_k,\sigma_k,\kappa_k), \quad \text{at } i=1 \text{ and } 2, \quad (23)$$

where, as before, μ_X and σ_X are the unknowns. The number $\widehat{\Psi}_{S_k}(s_i;\mu_k,\sigma_k,\kappa_k)$ consists entirely of known quantities and is evaluated only twice at s_1 and s_2 using (22), while $\widehat{\Psi}_Y(s_i;\mu_X,\sigma_X)$ is given by (9).

It can be seen that the mixture case, in which not all of the RVs follow the same type of distribution, can now be readily handled by using, as required, the corresponding expressions for the approximate MGFs for lognormal, lognormal-Rice, or Suzuki RVs.

VI. Numerical Examples

In the examples below, we plot the CDF and CCDF and use these results to provide guidelines on choosing generic values for s_1 and s_2 that work well in many cases. Small values of the CDF reveal the accuracy in tracking the head portion of the PDF, while small values of the CCDF reveal the accuracy in tracking the tail portion of the PDF.

A. Sum of Independent Lognormal RVs

Figure 2 plots the CDF and the CCDF of the sum of 6 independent lognormal RVs using Monte Carlo simulations, and compares it with the proposed method and the F-W and S-Y approximations. All the summands have a logarithmic variance of $\sigma=6$ dB and a mean of $\mu=0$ dB. It can be seen that the proposed method matches the head portion of the CDF very well when $(s_1,s_2)=(1.0,0.2)$ and is more accurate than both the F-W and the S-Y methods. While the S-Y method diverges from the actual CCDF in this scenario, the proposed method, for $(s_1,s_2)=(0.001,0.005)$, matches the simulation results well, and is similar to the F-W method in terms of accuracy. (The proposed method is more accurate for RV values below 400, while the F-W method is more accurate for RV values above 400.) We shall see that the same values of s_1 and s_2, used above, are accurate in several scenarios.

Figure 3 studies the accuracy of the approximation as the variance, σ, is varied from 4 dB to 12 dB. It shows the CDF for $K=6$ with $\mu=0$ dB for the summands. The effect of increasing the number of summands, K, is shown in Figure 4, which plots the CDF for different K. It can be seen from these two figures that $(s_1,s_2)=(1.0,0.2)$ again provides a good fit for various values of σ and K for approximating the head portion of the PDF. The F-W method is not shown due to its significant inaccuracy. It can be seen that the proposed method matches the simulation results well and is more accurate than the S-Y method. Similarly, $(s_1,s_2)=(0.001,0.005)$ is suitable for approximating the tail of the CCDF.

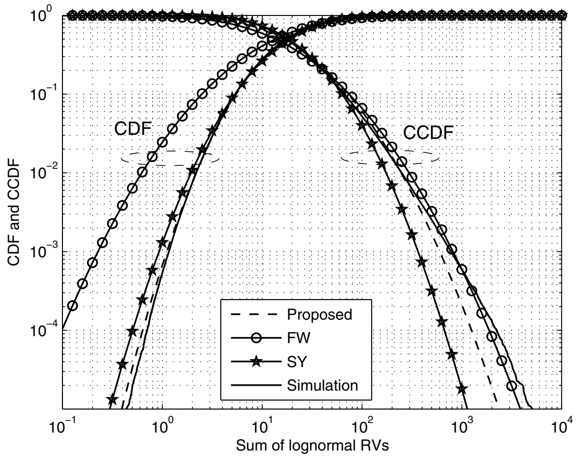


Fig. 2. Comparison of accuracy of CDF and CCDF computed using the F-W, S-Y, and proposed methods for approximating the sum of six independent lognormal RVs ($\sigma=6$ dB and $\mu=0$ dB).

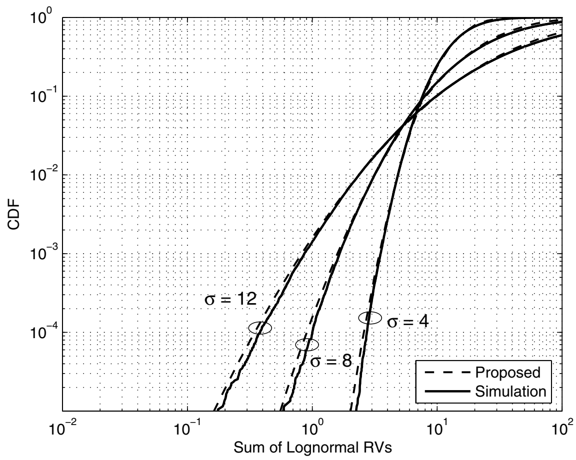


Fig. 3. Effect of variance, σ [dB], on the accuracy of approximating the CDF of the sum of independent lognormal RVs ($K=6$, $s_1=0.2$, $s_2=1.0$, $\mu=0$ dB).

B. Sum of Correlated Lognormal RVs

We now consider the sum of K correlated lognormal RVs, with the correlation matrix set as:

$$\mathbf{C}=\begin{bmatrix} 1 & \rho & \cdots & \rho^{K-1} \\ \rho & 1 & \cdots & \rho^{K-2} \\ & & \ddots & \\ \rho^{K-1} & \rho^{K-2} & \cdots & 1 \end{bmatrix}, \quad (24)$$

where ρ is the correlation coefficient between any two successive RVs. The logarithmic mean of the RVs is 0 dB.

The CDF obtained from the proposed technique is compared with simulation results in Figure 5 for the case of sum of

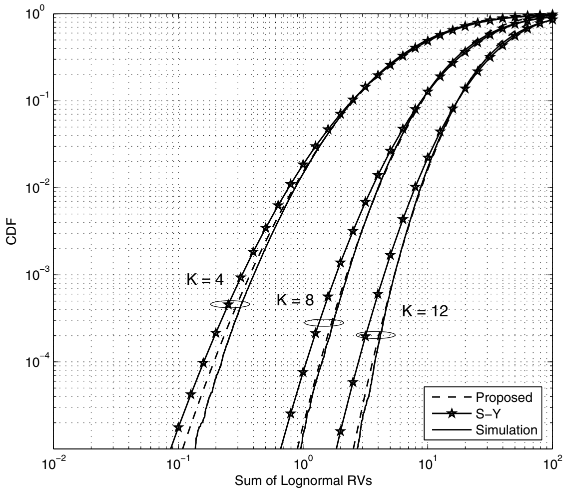


Fig. 4. Effect of number of summands, K, on the accuracy of approximating the CDF of the sum of independent lognormal RVs ($\sigma = 12$ dB, $\mu = 0$ dB). In all cases, $(s_1, s_2) = (1.0, 0.2)$. The F-W method is not shown due to its significant inaccuracy.

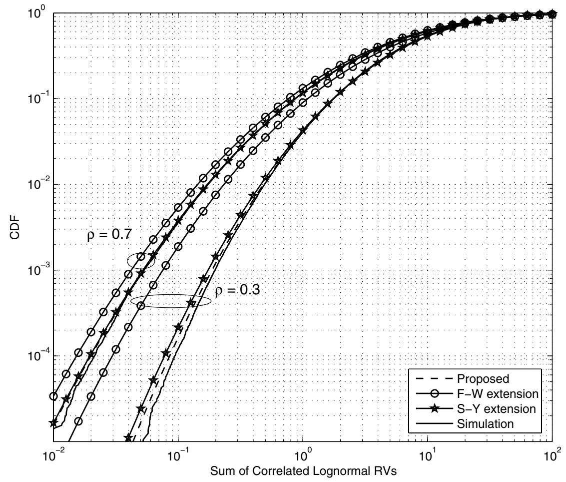


Fig. 5. Comparison of the accuracy of the techniques for the case of sum of correlated lognormal RVs for different correlation coefficients, ρ ($K = 4$, $\sigma = 8$ dB, and $\mu = 0$ dB). In all cases, $(s_1, s_2) = (1.0, 0.2)$.

four correlated lognormal RVs, each with $\sigma = 8$ dB and $\mu = 0$ dB. Also plotted are the CDFs from the F-W and S-Y extensions [24]. Two values of correlation coefficient are considered: $\rho = 0.3$ and $\rho = 0.7$. It can be seen that the proposed method can accurately track the CDF of the correlated lognormal sum, and is marginally better than the S-Y extension method. The F-W extension is the least accurate of all the methods. In case of the CCDF, the figure for which is not shown here, the accuracy of the proposed method is comparable to that of the F-W extension, and the S-Y extension is the least accurate. As expected, for larger correlation coefficients, all the methods can accurately track the CDF and the CCDF. The proposed method is also accurate when K is varied (figure not shown). As K decreases, the accuracy of the F-W and S-Y methods improves.

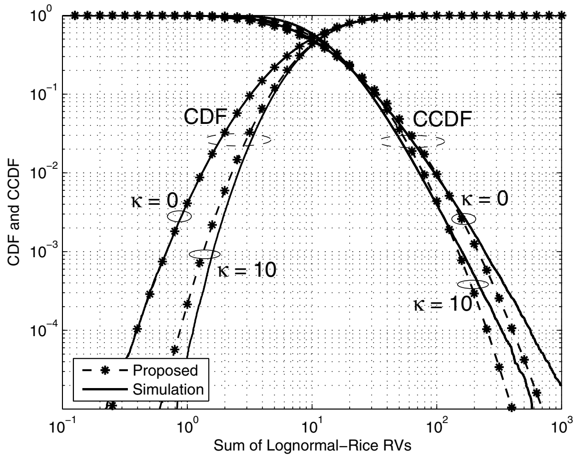


Fig. 6. Effect of Rice-coefficient (κ) on accuracy of approximating CDF and CCDF of the sum of lognormal-Rice RVs ($\sigma = 6$ dB, $K = 6$) ($s_1 = 0.2$, $s_2 = 1.0$ was used for CDF, and $s_1 = 0.001$, $s_2 = 0.005$ was used for CCDF).

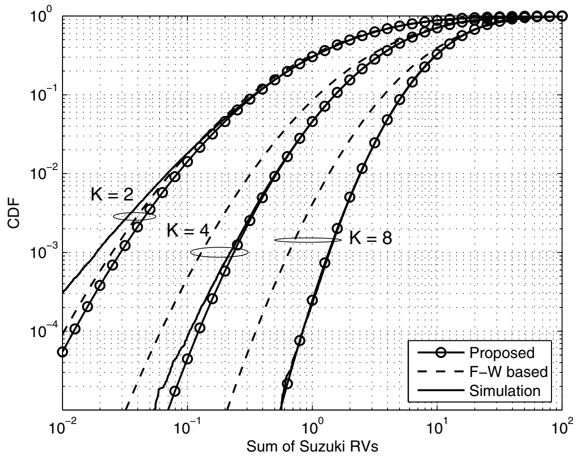


Fig. 7. Effect of number of Suzuki RVs on the accuracy of approximating the CDF for $\sigma = 6$ dB. (In all cases $s_1 = 0.2$, $s_2 = 1.0$).

C. Sum of Independent Suzuki and Lognormal-Rice RVs

The effect of the Rice-coefficient, κ, is examined in Figure 6, which plots the CDF and the CCDF of the sum of 6 lognormal-Rice RVs with a lognormal variance of 6 dB. We can see that both the CDF and the CCDF can be accurately approximated by the proposed method. The accuracy of the approximation improves as κ decreases.

Figure 7 plots the CDF of a sum of different numbers of independent Suzuki RVs using parameters obtained from (23) and compares them with Monte Carlo simulation results. It can be seen that the proposed method accurately approximates the sum of Suzuki RVs by a single lognormal RV. The result holds for $K = 2, 4$, and 8 summands.

VII. Accuracy in a Region of Interest and Sensitivity Analysis

We now quantitatively measure the accuracy and sensitivity of the method in a region of interest, in which the accuracy needs to be emphasized. For example, [6] uses the minimax criterion for the error in a region to fit its parameters, while Schleher's method advocates three parameter sets for three regions. We now show that the proposed method, with its two free parameters s_1 and s_2, provides the parametric flexibility to accurately model the behavior in a region of interest, for various parameter sets. This is done in this section using two common metrics that measure the relative deviation of the CDF or the CCDF curves in a region of interest.

Let $F^c_{(s_1,s_2)}(.)$ denote the CCDF and $F_{(s_1,s_2)}(.)$ denote the CDF of the lognormal distribution that approximates the sum of lognormal or lognormal-Rice RVs. Let H and H^c denote the empirically observed CDF and CCDF of the sum. These are obtained by Monte Carlo simulations. Let $y_1, \ldots, y_n$ denote n reference points in the region of interest. The accuracy metrics for CDF and CCDF are defined by:

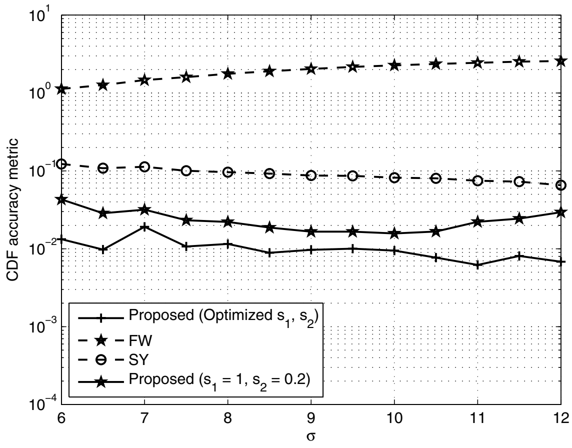
$$M_{\text{cdf}} = \sum_{i=1}^{R} e_i \frac{|H(y_i) - F_{(s_1,s_2)}(y_i)|}{H(y_i)}, \tag{25}$$

$$M_{\text{ccdf}} = \sum_{i=1}^{R} e_i^c \frac{|H^c(y_i) - F^c_{(s_1,s_2)}(y_i)|}{H^c(y_i)}, \tag{26}$$

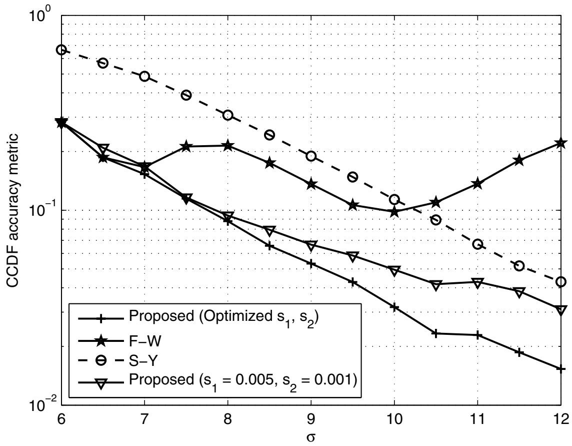
where e_i and e_i^c are the relative error weights for CDF and CCDF, respectively, to emphasize different accuracies in tracking different reference points. The weights are normalized such that $\sum_{i=1}^{R} e_i = 1$ and $\sum_{i=1}^{R} e_i^c = 1$.

The effect of σ on the accuracy possible in approximating the CDF and CCDF is studied in Figure 8. The error weights are set as $e_i = e_i^c = 1/R$, for all i. As an example, the region of interest for M_{cdf} is defined to be from $y_1 = 0$ dB to $y_R = 10$ dB, with the reference points spaced 1 dB apart ($R = 11$). The region of interest for M_{ccdf} is defined to be from 15 dB to 25 dB, with the reference points again spaced 1 dB apart. In Figure 8(a), the M_{cdf}, is plotted for the F-W and S-Y methods, and the proposed method. Two scenarios are considered for the proposed method. In the first scenario, the values of s_1 and s_2 are allowed to be optimized. This represents the best achievable accuracy of the proposed method for a given set of system parameters. An unconstrained Nelder-Mead non-linear maximization, easily implementable using Matlab's `fminsearch` function, was used for this purpose. In the second scenario, M_{cdf} achieved by the proposed method, when s_1 and s_2 are fixed at 1.0 and 0.2, is plotted. For the CDF metric, the S-Y method is, as expected, more accurate than the F-W method, while the proposed method with either fixed or optimized s_1 and s_2 values, is the most accurate of all methods for any given σ. As σ increases, the highest achievable accuracy of the proposed method and the S-Y method increases, while that of the F-W method decreases.

Similarly, figure 8(b) plots the CCDF accuracy metric, M_{ccdf}, for the three methods. As before, two scenarios for the proposed method are considered – when s_1 and s_2 are optimized to determine the best achievable accuracy and when s_1 and s_2 are fixed at 0.001 and 0.005. These values of s_1 and s_2 were used in several of the previous figures. As before, for any given σ, the proposed method, with fixed or optimized s_1 and s_2, is the most accurate. It can be seen that the highest achievable accuracy of the proposed method and the accuracy of the S-Y method improves as σ increases.



(a) CDF accuracy using M_{cdf}



(b) CCDF accuracy using M_{ccdf}

Fig. 8. Comparison of the accuracies, as a function of σ, of the proposed method and the F-W and S-Y methods. The range of interest is 0–10 dB, in steps of 1 dB, for M_{cdf} and is 15–25 dB, in steps of 1 dB, for M_{ccdf} ($K = 4$, $\mu = 0$ dB).

While the F-W method is as accurate as the proposed method for $\sigma \leq 6.5$ dB, it becomes the least accurate of the three methods for $\sigma > 10.2$ dB.

A. Sensitivity Analysis

Another topic of interest is the sensitivity of the proposed method to perturbations (or errors) in the MGF. These can arise due to the truncation errors, changes in the value of s_1 or s_2 or in the underlying parameter values, etc. A forward sensitivity analysis procedure [33], which measures the change in the accuracy metric of interest relative to a small change in the underlying MGFs, is provided in the Appendix. For simplicity, we focus on the case in which the lognormal RVs are independent and identically distributed. The analysis can be easily generalized to other cases. The results are shown graphically in Figure 9, which plots the upper bound on the sensitivity of the CDF accuracy metric (derived in the Appendix) as a function of the standard deviation, σ. Also plotted are results from simulations. It can be seen that the upper bound is loose for small σ, but becomes tighter for larger σ.

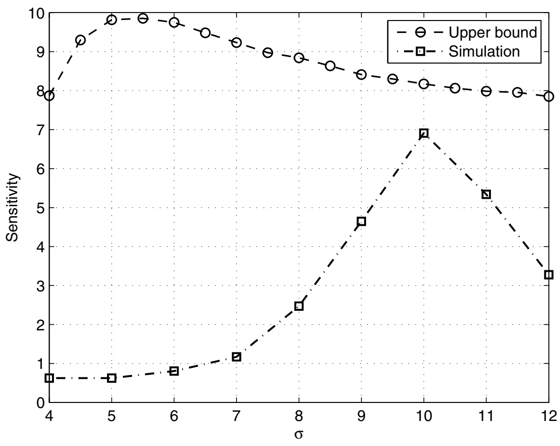


Fig. 9. Sensitivity of proposed method (measured by perturbations in the CDF accuracy metric) to small perturbations in the constituent MGFs ($K = 4$, $\mu = 0$ dB, and $(s_1, s_2) = (1.0, 0.2)$).

VIII. Conclusions

We proposed a simple and novel method to approximate the sum of several lognormal random variables with a single lognormal random variable. The method was motivated by the interpretation of the MGF as a weighted integral of the PDF. The MGF is a tool that provides the parametric flexibility needed to approximate, as accurately as required, different portions of the PDF. The method was shown to be general enough to cover the cases of independent (but not necessarily identical) lognormal RVs, arbitrarily correlated lognormal RVs, and independent lognormal-Rice and Suzuki RVs. It was shown to accurately model both the CDF and the CCDF of the lognormal sum distribution over a wide range of lognormal variances and means, and for different numbers of interferers. The proposed method was more accurate than the F-W and S-Y methods by one to two orders of magnitude. An upper bound for the sensitivity of the method, as measured by a perturbation in the accuracy metrics relative to a perturbation in the constituent MGFs, was also derived.

Appendix
Sensitivity Analysis

We are interested in evaluating the relative impact of a perturbation, δ, in the MGF of each of the lognormal RVs on the accuracy metric. From (25), it can be seen that the sensitivity, which is defined as $\lim_{\delta\to 0}\frac{|dM_{\text{cdf}}|}{\delta}$, is upper bounded by:

$$\lim_{\delta\to 0}\frac{|dM_{\text{cdf}}|}{\delta} \le \sum_{i=1}^{R}\frac{e_i}{H(y_i)}\lim_{\delta\to 0}\frac{|dF_{(s_1,s_2)}(y_i)|}{\delta}, \tag{27}$$

where $dF_{(s_1,s_2)}(y_i)$ is the perturbation in the CDF at y_i.

The CDF of lognormal approximation with mean μ_X dB and variance σ_X dB, which implicitly depend on the matching points s_1 and s_2, is $F_{(s_1,s_2)}(y_i) = 1 - Q\left(\frac{10\log_{10} y_i - \mu_X}{\sigma_X}\right)$. Therefore, for small δ, we have

$$\frac{|dF_{(s_1,s_2)}(y_i)|}{\delta} = \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left[-\frac{(10\log_{10} y_i - \mu_X)^2}{2\sigma_X^2}\right] \times\left|\sigma_X\frac{d\mu_X}{\delta} + (10\log_{10} y_i - \mu_X)\frac{d\sigma_X}{\delta}\right|. \tag{28}$$

From (10), it can be shown that the small perturbation δ in the MGF, $\widehat{\Psi}_{Y_k}(.;.,.)$, and the perturbations in values of μ_X and σ_X are related as follows:

$$\mathbf{G}\begin{bmatrix} d\mu_X/\delta \\ d\sigma_X/\delta \end{bmatrix} = K\begin{bmatrix} \widehat{\Psi}_{Y_k}(s_1;\mu_0,\sigma_0)^{K-1} \\ \widehat{\Psi}_{Y_k}(s_2;\mu_0,\sigma_0)^{K-1} \end{bmatrix}, \tag{29}$$

where $\mathbf{G} = -\begin{bmatrix} f_\mu(s_1) & f_\sigma(s_1) \\ f_\mu(s_2) & f_\sigma(s_2) \end{bmatrix}$ and

$$f_\sigma(s) = \frac{\sqrt{2}s}{\xi}\sum_{n=1}^{N}\frac{w_n a_n}{\sqrt{\pi}} \times\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi} - s\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi}\right)\right),$$

$$f_\mu(s) = \frac{s}{\xi}\sum_{n=1}^{N}\frac{w_n}{\sqrt{\pi}} \times\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi} - s\exp\left(\frac{\sqrt{2}\sigma_X a_n + \mu_X}{\xi}\right)\right).$$

Hence, the relative variations in the mean and standard deviation are given by

$$\begin{bmatrix} d\mu_X/\delta \\ d\sigma_X/\delta \end{bmatrix} = K\mathbf{G}^{-1}\begin{bmatrix} \widehat{\Psi}_{Y_k}(s_1;\mu_0,\sigma_0)^{K-1} \\ \widehat{\Psi}_{Y_k}(s_2;\mu_0,\sigma_0)^{K-1} \end{bmatrix}. \tag{30}$$

Combining the above equation with (28) and substituting in (27) yields the expression for the sensitivity.

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